Specifying Data Objects with Initial Algebras

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This study presents a systematic approach to specifying data objects with the help of initial algebras. The primary aim is to describe the set-up to be found in modern functional programming languages such as *Haskell* and *ML*, although it can also be applied to more general situations.

The 'initial algebra semantics' philosophy has been propagated by the ADJ group consisting of J.A. Goguen, J.W. Thatcher, E.G. Wagner and J.B. Wright, for example in [6], and is now well-established. The approach presented here can be seen as pushing this philosophy a stage further and consists of taking the following four steps.

(1) Data types are specified by a signature and the 'fully-defined' data objects are then described by the carrier sets in an initial algebra.

(2) The initial algebra is extended to include 'undefined' and 'partially defined' data objects, leading to what is known as a bottomed algebra. The correct bottomed algebra depends on the language being considered. However, it can always be defined to be an initial object in a class of bottomed algebras determined by a simple structure which we call a head type. This set-up can deal with both lazy and strict languages as well as anything in between.

(3) The third step is based on the observation that there is a unique family of partial orders defined on the carrier sets of the initial bottomed algebra such that each partial order can reasonably be interpreted as meaning 'being less-defined than'. This leads to what are called ordered algebras; these have partially ordered carrier sets and monotone operators. The initial bottomed algebra turns out to also be an initial object in an appropriate class of ordered algebras.

(4) The final step involves what is called the initial (or ideal) completion of a partially ordered set. This is used to complete the ordered algebra to end up with a continuous algebra having complete partially ordered carrier sets and continuous operators. The resulting continuous algebra is again an initial object in the appropriate class of continuous algebras.

The bottomed, ordered and continuous algebras which occur here are uniquely determined up to isomorphism by the signature and the head type, both of which are essentially finite structures in any practical case.

The account includes a treatment of polymorphism, although to simplify things the approach is a bit more restrictive than that to be found in languages such as *Haskell* or *ML*. Moreover, the polymorphic types here do not involve functional types, which means that data types such as lists whose components are functions are not allowed.

I have tried to keep the account self-contained, and have thus included all the standard results (together with their proofs) which are needed from universal

algebra and the theory of partially ordered sets. It is not assumed that the reader knows anything about functional programming, but some experience of this topic would, of course, not be amiss.

This account is an expanded version of part of *Computing with Equations* [14], some notes I wrote about the semantics of functional programming languages.

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Contents

Chapter 1

Introduction

In all modern functional programming languages the programmer can introduce new data types by what amounts to specifying a signature. Consider the following simple example of how a signature can be represented in most such languages:

$$\begin{aligned} \texttt{nat} &::= \texttt{Zero} \mid \texttt{Succ}\,\texttt{nat} \\ \texttt{pair} &::= \texttt{Pair}\,\texttt{nat}\,\texttt{nat} \\ \texttt{list} &::= \texttt{Nil} \mid \texttt{Cons}\,\texttt{nat}\,\texttt{list} \end{aligned}$$

This declares three new types with the names `nat`, `pair` and `list` together with five operator names `Zero`, `Succ`, `Pair`, `Nil` and `Cons`, each with its functionality (i.e., how many arguments it takes, the type of each argument and the type of the result, for instance `Cons` takes two arguments, the first of type `nat`, the second of type `list`, with `list` the type of the result).

Now the reason for giving a signature is because there is then a corresponding class of algebras. In the example an algebra is any collection of three sets $X_{\texttt{nat}}$, $X_{\texttt{pair}}$ and $X_{\texttt{list}}$ together with five mappings $p_{\texttt{Zero}} : \mathbb{I} \to X_{\texttt{nat}}$, $p_{\texttt{Succ}} : X_{\texttt{nat}} \to X_{\texttt{nat}}$, $p_{\texttt{Pair}} : X_{\texttt{nat}} \times X_{\texttt{nat}} \to X_{\texttt{pair}}$, $p_{\texttt{Nil}} : \mathbb{I} \to X_{\texttt{list}}$ and $p_{\texttt{Cons}} : X_{\texttt{nat}} \times X_{\texttt{list}} \to X_{\texttt{list}}$, where $\mathbb{I}$ is a set with one element (arising as an empty product) and where it should be clear how the domain and codomain of each mapping is determined by the functionality laid down in the signature. Such an algebra will be denoted for short just by (X, p).

Each algebra can be regarded as a realization of the signature. The sets $X_{\texttt{nat}}$, $X_{\texttt{pair}}$ and $X_{\texttt{list}}$ are respectively sets of data objects for the types `nat`, `pair` and `list`, the mappings $p_{\texttt{Zero}}$ and $p_{\texttt{Nil}}$ define constants (namely the single elements in their images) and the mappings $p_{\texttt{Succ}}$, $p_{\texttt{Pair}}$ and $p_{\texttt{Cons}}$ construct new data objects out of objects which have already been defined.

The problem is now to decide which algebra should be considered as the 'correct' realization of the signature. Of course, in the example the names `nat`, `pair`

and list suggest perhaps singling out the algebra with $X_{\mathtt{nat}} = \mathbb{N}$, $X_{\mathtt{pair}} = \mathbb{N}^2$ and $X_{\mathtt{list}} = \mathbb{N}^*$ (the set of all lists of elements from $\mathbb{N}$) and with $p_{\mathtt{Zero}}(\varepsilon) = 0$, $p_{\mathtt{Succ}}(n) = n+1$, $p_{\mathtt{Pair}}(m,n) = (m,n)$, $p_{\mathtt{Nil}}(\varepsilon) = \varepsilon$ and $p_{\mathtt{Cons}}(n,s) = n \triangleleft s$, where ε denotes both the single element in $\mathbb{I}$ and the empty list, and $n \triangleleft s$ is the operation of adding the element n to the beginning of the list s. But there are clearly many other algebras which look nothing like this particular choice, for instance with each occurrence of $\mathbb{N}$ replaced by $\mathbb{R}$, or with each of the sets $X_{\mathtt{nat}}$, $X_{\mathtt{pair}}$ and $X_{\mathtt{list}}$ consisting of just a single element.

However, there is really only one natural choice for the 'correct' realization and this is to take a so-called initial algebra. An algebra (X,p) is initial if for each algebra (X',p') there is a unique homomorphism from (X,p) to (X',p'), where a homomorphism is a structure-preserving family of mappings. In the example it would consist of three mappings $\pi_{\mathtt{nat}} : X_{\mathtt{nat}} \to X'_{\mathtt{nat}}$, $\pi_{\mathtt{pair}} : X_{\mathtt{pair}} \to X'_{\mathtt{pair}}$ and $\pi_{\mathtt{list}} : X_{\mathtt{list}} \to X'_{\mathtt{list}}$ satisfying five equations, one for each of the operator names with, for instance, the equation for `Cons` requiring that

$$\pi_{\mathtt{list}}(p_{\mathtt{Cons}}(x,s)) = p'_{\mathtt{Cons}}(\pi_{\mathtt{nat}}(x), \pi_{\mathtt{list}}(s))$$

should hold for all $x \in X_{\mathtt{nat}}$, $s \in X_{\mathtt{list}}$.

It turns out that an initial algebra exists for each signature (and the algebra singled out above is initial). Moreover, any two initial algebras are isomorphic and so an initial algebra is uniquely determined, up to isomorphism, by the signature. Furthermore, there is an explicit class of initial algebras, the term algebras, which are used in all functional programming languages to represent data objects. In a term algebra $(E,\square)$ the elements in the sets E_b are terms (or expressions), usually written in prefix form. For instance, in the example `Zero` and `Succ Succ Zero` are elements of $E_{\mathtt{nat}}$, `Pair Succ Zero Zero` an element of $E_{\mathtt{pair}}$ and `Nil` and `Cons Zero Cons Succ Zero Nil` elements of $E_{\mathtt{list}}$. (If the reader prefers to use braces to increase the legibility then the terms above with more then one element become `Succ (Succ Zero)`, `Pair (Succ Zero) Zero` and `Cons Zero (Cons (Succ Zero) Nil)`.)

Now since term algebras are initial, there is a unique isomorphism from a term algebra $(E,\square)$ to any other initial algebra, and in particular to an initial algebra (X,p) being used as the 'correct' realization of the signature. This means that each data object then has a unique representation as a term; for instance, in the algebra singled out above, the data object $3 \in \mathbb{N} = X_{\mathtt{nat}}$ is uniquely represented by the term `Succ Succ Succ Zero` $\in E_{\mathtt{nat}}$. The results about algebras, and in particular about initial algebras, which are needed to make all this precise are developed in Chapter 2.

The process outlined above is, however, only the first of several steps which are required to completely specify the data objects, and we now consider the second step. In any programming language data objects are manipulated by algorithms,

and in any non-trivial language it is an unavoidable fact that algorithms need not terminate. It is thus necessary to introduce an 'undefined' element for each type in order to represent this state of affairs. Moreover, depending on the language, it may also be necessary to have 'partially defined' data objects, for example an element of type `pair` in which the first component of a pair is defined but not the second, or an element of type `list` in which only some of the components in a list are defined. To deal with this situation bottomed algebras will be introduced. These are algebras containing for each type a special bottom element to denote an 'undefined' element of the type. (The bottom element will always be denoted by the symbol $\perp$, usually with a subscript to indicate which type is involved.)

The simplest way to obtain a bottomed algebra is to start with an ordinary algebra (X, p), add an 'undefined' element $\perp_b$ to X_b for each type b, and then extend each operator so that it produces an 'undefined' value as soon as one of its arguments is 'undefined'. This bottomed algebra is called the flat bottomed extension of (X, p), and by definition it does not contain any 'partially defined' objects.

The other extreme is an initial bottomed algebra, where a bottomed algebra (X, p) is initial if for each bottomed algebra (X', p') there is a unique bottomed homomorphism from (X, p) to (X', p'), and where a bottomed homomorphism is a homomorphism which maps bottom elements to bottom elements. As with ordinary algebras, there exist initial bottomed algebras and they are unique up to isomorphism. Moreover, in some sense they contain all possible 'partially defined' objects. One way to obtain an initial bottomed algebra is as a term algebra with the bottom elements added as constant terms. For the example this means that an initial bottomed algebra is given by the term algebra for the extended signature represented by

$$
\begin{aligned}
\texttt{nat} &::= \texttt{Zero} \mid \texttt{Succ}\ \texttt{nat} \mid \perp_{\texttt{nat}} \\
\texttt{pair} &::= \texttt{Pair}\ \texttt{nat}\ \texttt{nat} \mid \perp_{\texttt{pair}} \\
\texttt{list} &::= \texttt{Nil} \mid \texttt{Cons}\ \texttt{nat}\ \texttt{list} \mid \perp_{\texttt{list}}
\end{aligned}
$$

The 'correct' choice of a bottomed algebra depends on the application in hand. The semantics of most modern functional programming languages (such as *ML* or *Haskell*) require an initial bottomed algebra, whereas for a language such as *Lisp* the flat bottomed extension is the appropriate choice.

In Chapter 3 we develop a framework for dealing with bottomed algebras. There is a condition, called regularity, which plays an important role here, and requires that each element in the set $X_b \setminus \{\perp_b\}$ have a unique representation of the form $p_k(x_1, \ldots, x_n)$. In particular, regularity is needed in order to define the 'pattern matching' and 'case' operators which occur in all typed functional programming languages. We introduce the concept of a head type to deal with the various kinds of bottomed algebras which could arise when specifying the semantics of

a programming language. To each head type there is a corresponding class of bottomed algebras and this class possesses an initial element which is regular and is, as usual, unique up to isomorphism. This means that such an initial algebra is essentially uniquely determined by the signature and the head type. Moreover, the head types arising in practice are finite structures. This set-up includes both flat bottomed extensions and initial bottomed algebras as special cases.

The third step in our programme of completely specifying data objects arises from the observation that if (X, p) is a regular bottomed algebra then for each type b there is a unique partial order $\sqsubseteq_b$ on the set X_b such that $x \sqsubseteq_b x'$ can reasonably be interpreted as meaning that x is less-defined than x'. In particular, this should mean that $\perp_b \sqsubseteq x$ for each $x \in X_b$ and that each of the mappings p_k be monotone (using the corresponding product order on the domain of p_k).

These partial orders are essential when considering algorithms which manipulate data objects as dynamical systems. If an algorithm is designed to compute an element of type b then at each point of time there is an element of X_b which describes the present state of the computation. For any reasonable algorithm this state will be monotone as a function of time and should converge in some sense to the expected 'answer'. Now it is possible that an algorithm fails to terminate, not because it fails to produce an answer but because the answer is an infinite structure which cannot be computed completely in finite time. Typical examples here are algorithms which produce infinite lists, for instance a list consisting of all the prime numbers. In order to deal with this situation it is natural to complete the partially ordered set $(X_b, \sqsubseteq_b)$ for each type b to include all data objects which can arise as limits of finite computations. There is an appropriate completion which fits in with this interpretation and is called either the initial or the ideal completion of the partially ordered set.

The existence of the partial orders and the completion of the partially ordered sets are dealt with in Chapter 5. As preparation for this some general results about partial orders are first presented in Chapter 4.

There is an important aspect of specifying data objects which has not yet been discussed, and this is polymorphism. In the signature being used as an example there is a type `list` for lists having components of type `nat`. Now if lists having components of, say, type `pair` need to be implemented then the signature would have to be extended by adding a new type, say `listp`, together with two new operator names, say `Nilp` and `Consp`, whose functionalities are given by

$$\texttt{listp} ::= \texttt{Nilp} \mid \texttt{Consp pair listp}$$

Moreover, the signature has to be extended in essentially the same way with a new list type and two new operator names for each type for which lists are required. This is clearly not very satisfactory. What is needed is the possibility

of defining once and for all lists of an arbitrary type, and this is a feature of all modern functional programming languages. Our example signature could, for instance, be replaced by something like the following:

$$\begin{aligned}\texttt{nat} &::= \texttt{Zero} \mid \texttt{Succ nat}\\ \texttt{pair a b} &::= \texttt{Pair a b}\\ \texttt{list c} &::= \texttt{Nil} \mid \texttt{Cons c (list c)}\end{aligned}$$

The names `a`, `b` and `c` are called type variables and each can by replaced by a type to obtain a compound type. For example, there are the compound types `pair nat nat` and `list nat` corresponding to the types `pair` and `list` in the original signature. Arbitrarily complicated compound types can be generated, for example `pair (list (list nat)) (pair nat nat)`, which is a type whose objects are pairs with first component a list whose components are lists with components of type `nat`, and with second component a pair with both components of type `nat`. Signatures involving type variables and the corresponding algebras are treated in Chapter 6.

In order to give a foretaste of one of the main results in Chapter 2 and to make some of the concepts introduced above a bit more precise we end the Introduction by considering the following very simple signature

$$\texttt{nat} ::= \texttt{Zero} \mid \texttt{Succ nat}$$

In this case an algebra consists of just a set $X_{\texttt{nat}}$ together with two mappings $p_{\texttt{Zero}} : \mathbb{I} \to X_{\texttt{nat}}$ and $p_{\texttt{Succ}} : X_{\texttt{nat}} \to X_{\texttt{nat}}$, and is called a *natural number algebra*. It is convenient to represent such an algebra in the form $(X_{\texttt{nat}}, p_{\texttt{Nil}}(\varepsilon), p_{\texttt{Cons}})$, thus natural number algebras are exactly triples (X, e, p) consisting of a non-empty set X, an element $e \in X$ and a mapping $p : X \to X$. Of course, one such algebra is $(\mathbb{N}, 0, \mathsf{s})$, where $\mathsf{s}(n) = n+1$ for each $n \in \mathbb{N}$ (and hence the name natural number algebra). It can now be asked what is special about the algebra $(\mathbb{N}, 0, \mathsf{s})$.

To answer this question homomorphisms must be considered: If (X, e, p) and (X', e', p') are natural number algebras then a homomorphism is here a mapping $\pi : X \to X'$ such that $\pi(e) = e'$ and $p' \circ \pi = \pi \circ p$ (i.e., $p'(\pi(x)) = \pi(p(x))$ for all $x \in X$). Such a homomorphism π is an isomorphism (i.e., there exists a homomorphism π' from (X', e', p') to (X, e, p) with $\pi \circ \pi' = \mathrm{id}_{X'}$ and $\pi' \circ \pi = \mathrm{id}_X$) if and only if π is a bijection, and then π' is just the set-theoretic inverse π^{-1} of π. In this case (X, e, p) are (X', e', p') said to be *isomorphic*.

The identity mapping is clearly a homomorphism and the composition of two homomorphisms is again a homomorphism; being isomorphic thus defines an equivalence relation on the class of all natural number algebras. Therefore if the equivalence class containing $(\mathbb{N}, 0, \mathsf{s})$ can be identified in some reasonable way then the question asked above can be considered to have been answered satisfactorily. Two simple characterisations of this equivalence class are given below.

In the same way as before, a natural number algebra (X, e, p) is said to be *initial* if for each such algebra (X', e', p') there is a unique homomorphism from (X, e, p) to (X', e', p'). Then in particular $(\mathbb{N}, 0, \mathsf{s})$ is initial, since given (X', e', p'), a homomorphism from $(\mathbb{N}, 0, \mathsf{s})$ to (X', e', p') can be defined by induction, and its uniqueness also follows by induction. It follows that an algebra is isomorphic to $(\mathbb{N}, 0, \mathsf{s})$ if and only if it is initial. In other words, the equivalence class containing $(\mathbb{N}, 0, \mathsf{s})$ consists of exactly all the initial natural number algebras. This was the first characterisation.

Here is the second characterisation: A natural number algebra (X, e, p) will be called a *Peano triple* if the following three conditions are satisfied:

(1) The mapping p is injective.
(2) $p(x) \neq e$ for all $x \in X$.
(3) The only subset X' of X containing e with $p(x) \in X'$ for all $x \in X'$ is the set X itself.

Then $(\mathbb{N}, 0, \mathsf{s})$ is a Peano triple. (This is one of the possible formulations of the Peano axioms. In particular, the statement that $(\mathbb{N}, 0, \mathsf{s})$ satisfies (3) is nothing but the principle of mathematical induction.) The reader is left to show that an algebra is isomorphic to $(\mathbb{N}, 0, \mathsf{s})$ if and only if it is itself a Peano triple. In other words, the equivalence class containing $(\mathbb{N}, 0, \mathsf{s})$ also consists of exactly all the Peano triples.

Of course, a corollary of these two characterisations is that a natural number algebra is initial if and only if it is a Peano triple. This is a special case of an important result presented in Section 2.4 which states that an algebra is initial if and only if (in the terminology employed there) it is unambiguous and minimal. For a general algebra unambiguity corresponds to conditions (1) and (2) in the definition of a Peano triple, while minimality corresponds to condition (3). This characterisation is sometimes expressed by saying that initial algebras are exactly those for which there is *no confusion* (unambiguity) and *no junk* (minimality). The analysis of natural number algebras given here is essentially that to be found in Dedekind's book *Was sind und was sollen die Zahlen?* first published in 1888.

Chapter 2

Some universal algebra

The material presented in this chapter is all part of standard universal algebra. The classical field of universal algebra deals with the case of signatures having a single type, and so the only information required about each operator name is the number of arguments it takes. In this form the main problems were stated in Whitehead [17] and solved in Birkhoff [1]; standard texts are Cohn [4] and Grätzer [8]. The generalisation of the theory to *multi-sorted algebras* (i.e., to the algebras as they occur here) was made by Higgins [9] and Birkhoff and Lipson [3]. Birkhoff and Lipson (who speak of *heterogeneous algebras*) showed that essentially the whole of the classical theory carries over to the more general case.

Some of the first uses of multi-sorted algebras in computer science can be found in Maibaum [12] and Morris [13]. Their systematic use has been propagated by the ADJ group consisting of J.A. Goguen, J.W. Thatcher, E.G. Wagner and J.B. Wright, for example in the papers Goguen, Thatcher, Wagner and Wright [6] and Goguen, Thatcher and Wagner [7]. The emphasis here is very much on initial algebras in their various forms. The choice of material in this chapter is determined entirely by what will be needed later. The reader interested in a more balanced account of modern universal algebra should consult the books mentioned above.

2.1 Sets of various kinds

Before beginning in the next section with universal algebra proper the simple notions of a typing and of a bottomed set will be introduced here. Moreover, at the end of the section we say something about initial objects.

The first task, however, is to fix some notation. The empty set will be denoted by $\varnothing$ and the set $\{\varnothing\}$ by $\mathbb{I}$; thus $\mathbb{I}$ is the 'standard' set containing exactly one element. However, to avoid confusion the single element in $\mathbb{I}$ will be denoted by

ε (rather than by $\varnothing$). The set $\{\mathrm{T}, \mathrm{F}\}$ of boolean values will be denoted by $\mathbb{B}$. The set $\mathbb{N}$ of natural numbers is the set $\{0, 1, 2, \ldots\}$ (so 0 is considered to be a natural number). For each $n \in \mathbb{N}$ let $[n] = \{1, 2, \ldots, n\}$; in particular $[0] = \varnothing$.

The words *function* and *mapping* are considered to be synonyms. Let $f : X \to Y$ be a mapping; then X is called the *domain* of f and will be denoted by $\mathrm{dom}(f)$ and Y is called the *codomain*. For each $A \subset X$ put

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}$$

and for each $B \subset Y$ put $f^{-1}(B) = \{x \in X : f(x) \in B\}$. The subset $f(X)$ of the codomain Y is called the *image* of f and will be denoted by $\Im(f)$. If $f : X \to Y$ is a mapping and $A \subset X$ then $f_{|A}$ will be used to denote the *restriction* of f to A, thus $f_{|A} : A \to Y$ is the mapping given by $f_{|A}(x) = f(x)$ for all $x \in A$. If $f : X \to Y$ and $g : Y \to Z$ are mappings then their composition will be denoted by $g \circ f$, i.e., $g \circ f : X \to Z$ is the mapping defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

If X and Y are sets then Y^X will be used to denote the set of all mappings from X to Y. In particular, $Y^\varnothing = \mathbb{I}$ for each set Y. If $\alpha : X \to Y$ and J is a further set then α^J will be used to denote the induced mapping from X^J to Y^J defined by $\alpha^J(v) = \alpha \circ v$ for all $v \in X^J$.

Let S be a set and $\mathcal{C}$ be some class of objects (such as the class of all sets). Then by an S-*family* of objects from $\mathcal{C}$ is just meant a mapping $\alpha : S \to \mathcal{C}$. Such a family is said to be *finite* if the set S is finite. The usual convention for families will be followed in that the value of α applied to the argument s will be denoted by α_s rather than by $\alpha(s)$. If $\alpha : S \to \mathcal{C}$ is an S-family and $A \subset S$ then the restriction of α to A will be denoted (as for mappings) by $\alpha_{|A}$, thus $\alpha_{|A} : A \to \mathcal{C}$ is the A-family with $(\alpha_{|A})_a = \alpha_a$ for all $a \in A$.

It is useful to introduce some special notation for families of sets, i.e., for families $X : S \to \mathsf{Sets}$ with Sets the class of all sets. If X and Y are S-families of sets then $Y \subset X$ will mean that $Y_s \subset X_s$ for each $s \in S$. This will also be indicated by saying that X *contains* Y or that Y *is contained in* X. Moreover, $X \cap Y$ and $X \cup Y$ are the S-families of sets defined component-wise, i.e., $(X \cap Y)_s = X_s \cap Y_s$ and $(X \cup Y)_s = X_s \cup Y_s$ for each $s \in S$, and $\varnothing$ will be used to denote the S-family of sets with $\varnothing_s = \varnothing$ for each $s \in S$.

If X is a finite S-family of sets then $\otimes X$ will be used to denote the *cartesian product* of the sets in the family X: This is defined to be the set of all mappings $v : S \to \bigcup_{s \in S} X_s$ such that $v(s) \in X_s$ for each $s \in S$. (Of course, this definition makes sense when S is infinite, but we will only need finite products.) Note that $\otimes X = \mathbb{I}$ if $S = \varnothing$; moreover, if $Y \subset X$ then $\otimes Y$ can clearly be regarded as a subset of $\otimes X$.

Let $n \in \mathbb{N}$ and for each $j = 1, \ldots, n$ let X_j be a set; then there is the $[n]$-family X and thus the cartesian product $\otimes X$. This set will of course be denoted by

$X_1 \times \cdots \times X_n$; it is the set of all mappings ϱ from $[n]$ to $\bigcup_{j=1}^n X_j$ such that $\varrho(j) \in X_j$ for each j. In particular, if $n = 0$ then $X_1 \times \cdots \times X_n = \mathbb{I}$. For each j let $x_j \in X_j$; then as usual $(x_1, \ldots, x_n)$ denotes the the element ϱ of $X_1 \times \cdots \times X_n$ such that $\varrho(j) = x_j$ for each j; each element of $X_1 \times \cdots \times X_n$ has a unique representation of this form.

The simplest case of a cartesian product is when the sets $X_1, \ldots, X_n$ are all the same: Let X be a set; then for each $n \in \mathbb{N}$ the n-fold cartesian product of X with itself will be denoted by X^n, thus X^n is the set of all mappings from $[n]$ to X. In particular, $X^0 = \mathbb{I}$; moreover, it is convenient to identify X^1 with X in the obvious way.

For each set X the set $\bigcup_{n \geq 0} X^n$ will be denoted by X^*, which should be thought of as the set of all finite lists of elements from X. Note that since X^1 is being identified with X, it follows that X is a subset of X^*. (In other words, each element x of X is identified with the list whose single component is equal to x.) If $m \geq 1$ then the element $(x_1, \ldots, x_m)$ of $X^m \subset X^*$ will always be denoted simply by $x_1 \cdots x_m$.

If S is a set then by an S-family of mappings is meant a mapping $\varphi : S \to \mathsf{Maps}$ with Maps the class of all mappings between sets. Let $\varphi : S \to \mathsf{Maps}$ be such an S-family of mappings. Then for each $s \in S$ there exist sets X_s and Y_s such that $\varphi_s : X_s \to Y_s$ and in this case we write $\varphi : X \to Y$. More precisely, the statement that $\varphi : X \to Y$ is an S-family of mappings means that X and Y are S-families of sets and φ is an S-family of mappings with $\varphi_s : X_s \to Y_s$ for each $s \in S$. If Z is a further S-family of sets and $\psi : Y \to Z$ a further S-family of mappings then the S-family of composed mappings will be denoted by $\psi \circ \varphi$, thus $\psi \circ \varphi : X \to Z$ is the S-family of mappings with $(\psi \circ \varphi)_s = \psi_s \circ \varphi_s$ for each $s \in S$.

Now let S be a finite set, X and Y be S-families of sets and $\varphi : X \to Y$ be an S-family of mappings. Then there is a mapping $\otimes\varphi : \otimes X \to \otimes Y$ defined by

$$\otimes\varphi(v)(s) = \varphi_s(v(s))$$

for each $v \in \otimes X$, $s \in S$. Let $n \in \mathbb{N}$ and for each $j = 1, \ldots, n$ let $\varphi_j : X_j \to Y_j$ be a mapping; then $\otimes\varphi$ is just the mapping from $X_1 \times \cdots \times X_n$ to $Y_1 \times \cdots \times Y_n$ defined for each $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ by

$$\otimes\varphi(x_1, \ldots, x_n) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)) \ .$$

Lemma 2.1.1 *Let S be a finite set.*

(1) Let X be an S-family of sets and $\mathrm{id} : X \to X$ *be the S-family of identity mappings (i.e., with* $\mathrm{id}_s : X_s \to X_s$ *the identity mapping for each $s \in S$). Then* $\otimes\mathrm{id} : \otimes X \to \otimes X$ *is also the identity mapping.*

(2) Let X, Y and Z be S-families of sets and let $\varphi : X \to Y$ and $\psi : Y \to Z$ be S-families of mappings. Then $\otimes(\psi \circ \varphi) = \otimes\psi \circ \otimes\varphi$.

(3) Let X and Y be S-families of sets and let $\varphi : X \to Y$ be an S-family of mappings. If the mapping $\varphi_s : X_s \to Y_s$ is injective (resp. surjective) for each $s \in S$ then the mapping $\otimes\varphi : \otimes X \to \otimes Y$ is also injective (resp. surjective). In particular, if $\varphi_s : X_s \to Y_s$ is a bijection for each $s \in S$ then the mapping $\otimes\varphi$ is a bijection, and in this case $(\otimes\varphi)^{-1} = \otimes\varphi^{-1}$, where φ^{-1} is the S-family of mappings with $(\varphi^{-1})_s = (\varphi_s)^{-1}$ for each $s \in S$.

Proof (1) This is clear.

(2) Let $v \in \otimes X$ and $s \in S$; then

$$\begin{aligned} \otimes(\psi \circ \varphi)(v)(s) &= (\psi_s \circ \varphi_s)(v(s)) = \psi_s(\varphi_s(v(s))) \\ &= \psi_s(\otimes\varphi(v))(s) = \otimes\psi(\otimes\varphi(v))(s) = (\otimes\psi \circ \otimes\varphi)(v)(s) \end{aligned}$$

i.e., $\otimes(\psi \circ \varphi) = \otimes\psi \circ \otimes\varphi$.

(3) If φ_s is injective for each s then there is an S-family of mappings $\psi : Y \to X$ such that $\psi_s \circ \varphi_s$ is the identity mapping on X_s for each $s \in S$. Then by (1) $\otimes(\psi \circ \varphi)$ is the identity mapping on $\otimes X$ and by (2) $\otimes\psi \circ \otimes\varphi = \otimes(\psi \circ \varphi)$. Hence $\otimes\varphi$ is injective. The other case is almost identical: If φ_s is surjective for each s then there exists an S-family of mappings $\psi : Y \to X$ such that $\varphi_s \circ \psi_s$ is the identity mapping on Y_s for each $s \in S$, and as in the first part it then follows that $\otimes\varphi \circ \otimes\psi$ is the identity mapping on $\otimes Y$. Finally, if φ_s is a bijection for each $s \in S$ then $\otimes\varphi^{-1} \circ \otimes\varphi$ is the identity mapping on $\otimes X$ and $\otimes\varphi \circ \otimes\varphi^{-1}$ is the identity mapping on $\otimes Y$, and thus $(\otimes\varphi)^{-1} = \otimes\varphi^{-1}$. □

We now come to the notion of a typing. This takes into account the situation met with in most modern programming languages in which each name occurring in a program is assigned, either explicitly or implicitly, a type. The kind of object a name can refer to is then determined by its type. If S is a set then a mapping $\gamma : I \to S$ will be called an S-*typing*. The set S should here be thought of as a set of 'types', I as a set of 'names' and γ as specifying the type of objects to which the 'names' can be assigned. If the set I is finite then γ is called a *finite* S-typing and the class of all finite S-typings will be denoted by $\mathcal{T}(S)$. In accordance with our previous notation the unique S-typing γ with $\mathrm{dom}(\gamma) = \varnothing$ will be denoted by ε. Note that there is an obvious one-to-one correspondence between $\mathbb{I}$-typings

and sets, since for each set I there is a unique mapping from I to $\mathbb{I}$. In particular, $\mathcal{T}(\mathbb{I})$ can be considered as the class FSets of all finite sets.

If $\gamma : I \to S$ is an S-typing then the pair (I, γ) will be referred to as an *S-typed set*. Of course, there is no real difference between S-typings and S-typed sets. We will work mainly with S-typings because this tends to result in simpler notation.

Let $\alpha : S \to \mathcal{C}$ be an S-family and $\gamma : I \to S$ be an S-typing. Then there is an I-family $\alpha \circ \gamma : I \to \mathcal{C}$, (which means that $(\alpha \circ \gamma)_\eta = \alpha_{\gamma(\eta)}$ for each $\eta \in I$). If X is an S-family of sets and $\gamma : I \to S$ a finite S-typing then the product $\otimes(X \circ \gamma)$ will be denoted by X^γ; thus X^γ is the set of all *typed* mappings from I to $\bigcup_{\eta \in I} X_{\gamma(\eta)}$, i.e., the set of mappings $v : I \to \bigcup_{\eta \in I} X_{\gamma(\eta)}$ such that $v(\eta) \in X_{\gamma(\eta)}$ for each $\eta \in I$. The elements of X^γ are called *assignments*. An assignment $v \in X^\gamma$ thus assigns to each 'name' $\eta \in I$ an element $v(\eta) \in X_{\gamma(\eta)}$ of the appropriate type. Note that $X^\varepsilon = \mathbb{I}$; moreover, if $Y \subset X$ then Y^γ can clearly be regarded as a subset of X^γ.

The elements of S^* will be considered as finite S-typings (and so S^* will be regarded as a subset of $\mathcal{T}(S)$): The list $s_1 \cdots s_n$ is identified with the mapping from $[n]$ to S which assigns to each j the type s_j. (The list ε with no components is then the S-typing ε.) Let X_S be a family of sets and $\sigma = s_1 \cdots s_n \in S^*$; then, considering σ as an S-typed set, $X^\sigma = Y_1 \times \cdots \times Y_n$ with $Y_j = X_{s_j}$ for each j, which leads to the usual notation $X_{s_1} \times \cdots \times X_{s_n}$ for the set X^σ.

Lemma 2.1.2 *Let X, Y and Z be S-families of sets and let $\varphi : X \to Y$ and $\psi : Y \to Z$ be S-families of mappings. Then $(\psi \circ \varphi) \circ \gamma = (\psi \circ \gamma) \circ (\varphi \circ \gamma)$ for each S-typing γ.*

Proof This holds since for each $\eta \in \mathrm{dom}(\gamma)$

$$((\psi \circ \varphi) \circ \gamma)_\eta = (\psi \circ \varphi)_{\gamma(\eta)} = \psi_{\gamma(\eta)} \circ \varphi_{\gamma(\eta)} = (\psi \circ \gamma)_\eta \circ (\varphi \circ \gamma)_\eta = ((\psi \circ \gamma) \circ (\varphi \circ \gamma))_\eta \,. \; \square$$

Now let X and Y be S-families of sets and let $\varphi : X \to Y$ be an S-family of mappings. Then for each finite S-typing γ the mapping $\otimes(\varphi \circ \gamma)$ will denoted by φ^γ. Thus $\varphi^\gamma : X^\gamma \to Y^\gamma$ is the mapping given by

$$\varphi^\gamma(v)(\eta) = \varphi_{\gamma(\eta)}(v(\eta))$$

for each $v \in X^\gamma$, $\eta \in \mathrm{dom}(\gamma)$. Note that if $\sigma = s_1 \cdots s_n \in S^*$ then φ^σ is just the mapping from $X^\sigma = X_{s_1} \times \cdots \times X_{s_n}$ to $Y^\sigma = Y_{s_1} \times \cdots \times Y_{s_n}$ given by

$$\varphi^\sigma(x_1, \ldots, x_n) = (\varphi_{s_1}(x_1), \ldots, \varphi_{s_n}(x_n))$$

for each $(x_1, \ldots, x_n) \in X_{s_1} \times \cdots \times X_{s_n}$.

Lemma 2.1.3 *(1) Let X be an S-family of sets and* $\mathrm{id} : X \to X$ *the S-family of identity mappings. Then* $\mathrm{id}^\gamma : X^\gamma \to X^\gamma$ *is also the identity mapping for each finite S-typing γ.*

(2) Let X, Y and Z be S-families of sets and let $\varphi : X \to Y$ and $\psi : Y \to Z$ be S-families of mappings. Then $(\psi \circ \varphi)^\gamma = \psi^\gamma \circ \varphi^\gamma$ for each finite S-typing γ.

(3) Let X and Y be S-families of sets and let $\varphi : X \to Y$ be an S-family of mappings. If the mapping $\varphi_s : X_s \to Y_s$ is injective (resp. surjective) for each $s \in S$ then for each finite S-typing γ the mapping $\varphi^\gamma : X^\gamma \to Y^\gamma$ is also injective (resp. surjective). In particular, if $\varphi_s : X_s \to Y_s$ is a bijection for each $s \in S$ then the mapping φ^γ is a bijection, and in this case $(\varphi^\gamma)^{-1} = (\varphi^{-1})^\gamma$.

Proof (1) is clear; moreover, by Lemma 2.1.1 (2) and Lemma 2.1.2

$$(\psi \circ \varphi)^\gamma = \otimes((\psi \circ \varphi) \circ \gamma) = \otimes((\psi \circ \gamma) \circ (\varphi \circ \gamma)) = \otimes(\psi \circ \gamma) \circ \otimes(\varphi \circ \gamma) = \psi^\gamma \circ \varphi^\gamma ,$$

which is (2), and (3) follows exactly as in Lemma 2.1.1 (3). □

The notion of a $\mathcal{C}$-typing still makes sense when $\mathcal{C}$ is a class. In particular, $\mathcal{T}(\mathcal{C})$ will be used to denote the class of all finite $\mathcal{C}$-typings. Note, however, that a $\mathcal{C}$-typing $\gamma : I \to \mathcal{C}$ is exactly an I-family of objects from $\mathcal{C}$, and so $\mathcal{T}(\mathcal{C})$ is also the class of all finite families of objects from $\mathcal{C}$.

We now come to the second simple notion, that of a bottomed set, which will play an important role. A *bottomed set* is a pair $(X, \perp)$ consisting of a set X and a distinguished *bottom element* $\perp \in X$. The bottom element $\perp$ should be thought of as representing an 'undefined' or 'completely unknown' value which, if it were to be 'more defined' or 'become more known', would take on the value of one of the remaining elements of X.

If $(X, \perp)$ and $(X', \perp')$ are bottomed sets then by a mapping $f : (X, \perp) \to (X', \perp')$ is just meant a mapping $f : X \to X'$. Such a mapping is said to be *bottomed* (or *strict*) if $f(\perp) = \perp'$. It should be noted, however, that the mappings between bottomed sets which will occur here are typically not bottomed. A bottomed mapping $f : X \to X'$ will be called *proper* if $f(x) \neq \perp'$ for all $x \in X \setminus \{\perp\}$.

If $(X, \perp)$ and $(X', \perp')$ are bottomed sets then $(X, \perp) \subset (X', \perp')$ will mean that $X \subset X'$ and $\perp = \perp'$.

If $(X, \perp)$ is a bottomed set then it is usual to write just X instead of $(X, \perp)$ and to assume that the bottom element $\perp$ can be inferred from the context. The set X will be referred to as the *underlying set* if it is necessary to distinguish it from the bottomed set X. If X is a bottomed set then the set $X \setminus \{\perp\}$ will be denoted by $X^\natural$.

Note that if X and X' are bottomed sets then $X \subset X'$ means that $X \subset X'$ (as sets) and that X and X' have a common bottom element.

Let X be a finite S-family of bottomed sets with $\perp_s$ the bottom element of X_s for each $s \in S$. Then the product set $\otimes X$ (defined in terms of the S-family of sets X) will be considered as a bottomed set by stipulating that its bottom element $\perp$ be the mapping given by $\perp(s) = \perp_s$ for each $s \in S$. A special case of this is when $n \geq 2$ and X_j is a bottomed set with bottom element $\perp_j$ for each $j = 1, \ldots, n$; then the product $X_1 \times \cdots \times X_n$ is considered as a bottomed set with bottom element $(\perp_1, \ldots, \perp_n)$. Note that this definition implies in particular that $\mathbb{I}$ is considered to be a bottomed set (with of course ε as bottom element).

If S is an arbitrary set, X an S-family of bottomed sets and γ a finite S-typing then the bottomed set $\otimes(X \circ \gamma)$ will be denoted by X^γ. Thus the bottomed set X^γ is just the set X^γ together with the bottom element $\perp$ given by $\perp(\eta) = \perp_{\gamma(\eta)}$ for each $\eta \in \mathrm{dom}(\gamma)$.

Almost all the constructions to be made in these notes result in what are called *initial objects* and there is one trivial property of such objects which is worth noting, namely that if they exist in a given category then they are isomorphic. Isolating this fact here helps to avoid repeating the same kind of argument for each special case. Now since the concept of being initial involves a category, we must first say what this is: A *category* C consists of

— a class of elements $\mathcal{C}$ called the *objects* of the category,

— for each $X, Y \in \mathcal{C}$ a set $\mathrm{Hom}(X,Y)$, whose elements are called *morphisms* with *domain* X and *codomain* Y,

— for each $X, Y, Z \in \mathcal{C}$ a mapping $(f,g) \mapsto g \circ f$ from $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z)$ to $\mathrm{Hom}(X,Z)$,

such that the following two conditions hold:

— *(Associativity)* If $f \in \mathrm{Hom}(W,X)$, $g \in \mathrm{Hom}(X,Y)$ and $h \in \mathrm{Hom}(Y,Z)$ then

$$(h \circ g) \circ f = h \circ (g \circ f) .$$

— *(Identity)* For each $X \in \mathcal{C}$ there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for all $f \in \mathrm{Hom}(X,Y)$, $g \in \mathrm{Hom}(Y,X)$ and all $Y \in \mathcal{C}$.

For each $X \in \mathcal{C}$ the morphism id_X is unique: If $\mathrm{id}'_X \in \mathrm{Hom}(X,X)$ is a further morphism with this property then $\mathrm{id}'_X = \mathrm{id}'_X \circ \mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}_X$.

In the categories we will be dealing with the composition of morphisms $\circ$ is always some kind of composition of mappings (or families of mappings). This means that the associativity of $\circ$ is a trivial consequence of the fact that the composition of mappings is associative. Similarly, the identity morphisms will always be identity mappings (or families of identity mappings), and so they will trivially satisfy the defining condition.

The simplest example of a category is the the category of sets, i.e., the category with $\mathcal{C} = \mathsf{Sets}$, with $\mathrm{Hom}(X, Y)$ the set of all mappings from X to Y for each X, $Y \in \mathcal{C}$, and with $\circ$ the usual composition of mappings. This category will be denoted (along with its objects) by Sets.

A second example is the category of bottomed sets, i.e., the category with $\mathcal{C}$ the class of all bottomed sets, with $\mathrm{Hom}(X, Y)$ the set of all mappings from X to Y for each X, $Y \in \mathcal{C}$, and again with $\circ$ the usual composition of mappings. This category will be denoted (along with its objects) by BSets.

Note that in both these categories there is a mapping $\otimes : \mathcal{T}(\mathcal{C}) \to \mathcal{C}$ which gives the product of each finite family of objects. Moreover, if S is an arbitrary set, X an S-family of objects from $\mathcal{C}$ and γ a finite S-typing then in both Sets and BSets the object $\otimes(X \circ \gamma)$ is being denoted by X^γ.

Let C be a category; a morphism $f \in \mathrm{Hom}(X, Y)$ is called an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In this case the inverse g is uniquely determined by f. Objects X, $Y \in \mathcal{C}$ are said to be *isomorphic* if there exists an isomorphism $f \in \mathrm{Hom}(X, Y)$. It is easy to see that being isomorphic defines an equivalence relation on the class $\mathcal{C}$.

An object $X \in \mathcal{C}$ is said to be *initial* if for each object $Y \in \mathcal{C}$ there exists a unique morphism $f : X \to Y$, i.e., if the set $\mathrm{Hom}(X, Y)$ consists of a single element for each $Y \in \mathcal{C}$. In particular, it must then be the case that id_X is the only element in $\mathrm{Hom}(X, X)$. Initial objects need not exist in a category C: For instance, they do not exist in the category BSets (since morphisms are not necessarily strict mappings). Moreover, they can exist but be rather trivial: In the category Sets the empty set $\varnothing$ is the only initial object. However, for the categories involving various kinds of algebras which we will be dealing with there are non-trivial initial objects, and the following result shows they are then unique up to isomorphism:

Proposition 2.1.1 *Initial objects X, $Y \in \mathcal{C}$ are isomorphic. Conversely, if X is an initial object and Y is isomorphic to X then Y is also initial.*

Proof Suppose first that X and Y are initial objects. In particular, id_X is the only element in $\mathrm{Hom}(X, X)$ and id_Y is the only element in $\mathrm{Hom}(Y, Y)$. Moreover, there exists a unique morphism $f : X \to Y$ and a unique morphism $f : Y \to X$. Thus $g \circ f \in \mathrm{Hom}(X, X)$ and so $g \circ f = \mathrm{id}_X$, and in the same way $f \circ g = \mathrm{id}_Y$. Hence f is an isomorphism and X and Y are isomorphic.

Conversely, let $X \in \mathcal{C}$ be an initial object and Y be isomorphic to X. Since X and Y are isomorphic there exist morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Let $Z \in \mathcal{C}$; since X is initial there exists a (unique) morphism $h : X \to Z$ and then $h \circ g \in \mathrm{Hom}(Y, Z)$. Consider h_1, $h_2 \in \mathrm{Hom}(Y, Z)$;

then $h_1 \circ g$ and $h_2 \circ g$ are both elements of $\mathrm{Hom}(X, Z)$ and hence $h_1 \circ g = h_2 \circ g$, since X is initial. It thus follows that

$$h_1 = h_1 \circ (g \circ f) = (h_1 \circ g) \circ f = (h_2 \circ g) \circ f = h_2 \circ (g \circ f) = h_2 \ ,$$

which shows there exists a unique morphism in $\mathrm{Hom}(Y, Z)$ for each object Z. $\square$

The statement in Proposition 2.1.1 is an example of what Lang would refer to as 'abstract nonsense'. However, it should be borne in mind each time an initial object arises in what follows. Here is further piece of 'abstract nonsense' which will be applied a couple of times:

Proposition 2.1.2 *Let P be a property of the objects C satisfying the following three conditions:*

(1) There exists an object having property P.

(2) Every object having property P is initial.

(3) Any object isomorphic to an object having property P also has property P.

Then each initial object has property P, and so having property P is equivalent to being initial.

Proof Let X be an initial object; by (1) there exists an object Y having property P, and by (2) and Proposition 2.1.1 X and Y are isomorphic; thus by (3) X has property P. $\square$

A category C' is a *subcategory* of a category C if

— the objects of C' are a subclass of the objects of C,

— for all objects X, Y of C' the set of morphisms $\mathrm{Hom}'(X, Y)$ in C' is a subset of the set of morphisms $\mathrm{Hom}(X, Y)$ in C,

— the composition of morphisms in C' is the restriction of the composition of morphisms in C.

C' is called a *full subcategory* of C if $\mathrm{Hom}'(X, Y) = \mathrm{Hom}(X, Y)$ for all objects X, Y of C'. Given C, such a full subcategory is determined by specifying a subclass of the objects of C.

Let us emphasise that categories play a very superficial role here, and are needed simply to formalise what it means to be initial. Readers not familiar with this kind of stuff are recommended to look in Mac Lane and Birkhoff's Algebra text [11].

2.2 Algebras and homomorphisms

The structure which plays a fundamental role in all of what follows is that of an algebra associated with a signature. What is usually called a signature we will call an *enumerated signature*. This is a triple $\Lambda = (B, K, \Theta)$, where B and K are non-empty sets and $\Theta : K \to B^* \times B$ is a mapping. The set B should here be thought of as a set of *types*, K can be regarded as a set of *operator names*, and for each $k \in K$ the pair $\Theta(k)$ specifies the type of the domain and the codomain of the operator named by k. If $k \in K$ with $\Theta(k) = (b_1 \cdots b_n, b)$ then we say that *k has type* $b_1 \cdots b_n \to b$.

If $\Lambda = (B, K, \Theta)$ is an enumerated signature then a Λ-*algebra* is any pair (X, p) consisting of a B-family of sets X and a K-family of mappings p such that p_k is a mapping from $X_{b_1} \times \cdots \times X_{b_n}$ to X_b whenever $k \in K$ is of type $b_1 \cdots b_n \to b$. For each $b \in B$ the set X_b should be thought of as a set of elements of type b and for each $k \in K$ the mapping p_k can be thought of as the operator corresponding to the operator name k.

A simple enumerated signature $\Lambda = (B, K, \Theta)$ with a corresponding 'natural' Λ-algebra (X, p) are given in Example 2.2.1 on the following page. The usual way of representing such a signature is then illustrated in Example 2.2.2.

Recall that $\mathcal{T}(B)$ denotes the class of all finite B-typings and that B^* is considered to be a subset of $\mathcal{T}(B)$ in that $\sigma = b_1 \cdots b_n \in B^*$ is identified with the mapping from $[n]$ to B which assigns j the type b_j for each $j = 1, \ldots, n$. Our definition of a signature is obtained by replacing B^* with $\mathcal{T}(B)$ in the definition of an enumerated signature. This means that a *signature* is a triple $\Lambda = (B, K, \Theta)$ consisting of non-empty sets B and K and a mapping $\Theta : K \to \mathcal{T}(B) \times B$.

The two components of Θ will be denoted by $\Theta^{\triangleright}$ and $\Theta_{\triangleleft}$, thus $\Theta^{\triangleright} : K \to \mathcal{T}(B)$, $\Theta_{\triangleleft} : K \to B$, and $\Theta(k) = (\Theta^{\triangleright}(k), \Theta_{\triangleleft}(k))$ for each $k \in K$. However, we almost always just write $k^{\triangleright}$ instead of $\Theta^{\triangleright}(k)$ and $k_{\triangleleft}$ instead of $\Theta_{\triangleleft}(k)$. Moreover, for each $k \in K$ the set $\mathrm{dom}(k^{\triangleright})$ will be denoted by $\langle k^{\triangleright} \rangle$, so $k^{\triangleright} : \langle k^{\triangleright} \rangle \to B$ and $k^{\triangleright}\eta$ will be used as an alternative notation for $k^{\triangleright}(\eta)$ for each $\eta \in \langle k^{\triangleright} \rangle$.

If $\Lambda = (B, K, \Theta)$ is a signature then a Λ-*algebra* is any pair (X, p) consisting of a B-family of sets X and a K-family of mappings p such that p_k is a mapping from $X^{k^{\triangleright}}$ to $X_{k_{\triangleleft}}$ for each $k \in K$. For enumerated signatures this gives the same definition as above, since if $\sigma = b_1 \cdots b_n \in B^*$ then $X^{\sigma} = X_{b_1} \times \cdots \times X_{b_n}$.

The reason for working with the more general definition is that in the long run it turns out to be more natural. However, the reader not wanting to believe this can simply assume that all the signatures occurring are enumerated. In this case $k^{\triangleright}$ is always an element $b_1 \cdots b_n$ of B^* and $X^{k^{\triangleright}}$ should just be thought of as a compact way of denoting the product $X_{b_1} \times \cdots \times X_{b_n}$. Moreover, $\langle k^{\triangleright} \rangle$ is then the set $[n]$ and $k^{\triangleright}\eta = b_\eta$ for each $\eta \in [n]$.

It is always possible to replace a general signature with an 'equivalent' enumerated signature. (This really just amounts to fixing an enumeration of the elements in the set $\langle k^{\triangleright}\rangle$ for each $k \in K$.)

Example 2.2.1 $\underline{Int}$ will always be used to denote the subset of the set $\{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3},\mathtt{4},\mathtt{5},\mathtt{6},\mathtt{7},\mathtt{8},\mathtt{9},\mathtt{-}\}^*$ containing for each integer n its standard representation $\underline{n}$ as a string of characters. The mapping $n \mapsto \underline{n}$ thus maps $\mathbb{Z}$ bijectively onto $\underline{Int}$.

Define an enumerated signature $\Lambda = (B, K, \Theta)$ by letting

$B = \{\mathtt{bool}, \mathtt{nat}, \mathtt{int}, \mathtt{pair}, \mathtt{list}\}$,

$K = \{\mathtt{True}, \mathtt{False}, \mathtt{Zero}, \mathtt{Succ}, \mathtt{Pair}, \mathtt{Nil}, \mathtt{Cons}\} \cup \underline{Int}$,

and with $\Theta : K \to B^* \times B$ defined by

$\Theta(\mathtt{True}) = \Theta(\mathtt{False}) = (\varepsilon, \mathtt{bool})$,

$\Theta(\mathtt{Zero}) = (\varepsilon, \mathtt{nat})$, $\Theta(\mathtt{Succ}) = (\mathtt{nat}, \mathtt{nat})$,

$\Theta(\mathtt{Pair}) = (\mathtt{int}\ \mathtt{int}, \mathtt{int})$,

$\Theta(\mathtt{Nil}) = (\varepsilon, \mathtt{list})$, $\Theta(\mathtt{Cons}) = (\mathtt{int}\ \mathtt{list}, \mathtt{list})$,

$\Theta(\underline{n}) = (\varepsilon, \mathtt{int})$ for each $n \in \mathbb{Z}$.

This means that $\mathtt{True}$ and $\mathtt{False}$ are of type $\varepsilon \to \mathtt{bool}$, $\mathtt{Zero}$ is of type $\varepsilon \to \mathtt{nat}$, $\mathtt{Succ}$ of type $\mathtt{nat} \to \mathtt{nat}$, $\mathtt{Pair}$ of type $\mathtt{int}\ \mathtt{int} \to \mathtt{pair}$, $\mathtt{Nil}$ of type $\varepsilon \to \mathtt{list}$, $\mathtt{Cons}$ of type $\mathtt{int}\ \mathtt{list} \to \mathtt{list}$, and $\underline{n}$ is of type $\varepsilon \to \mathtt{int}$ for each $n \in \mathbb{Z}$.

Now define a Λ-algebra (X, p) with a B-family of sets X and a K-family of mappings p by letting

$X_{\mathtt{bool}} = \mathbb{B}$, $X_{\mathtt{nat}} = \mathbb{N}$, $X_{\mathtt{int}} = \mathbb{Z}$, $X_{\mathtt{pair}} = \mathbb{Z}^2$, $X_{\mathtt{list}} = \mathbb{Z}^*$,

$p_{\mathtt{True}} : \mathbb{I} \to X_{\mathtt{bool}}$ with $p_{\mathtt{True}}(\varepsilon) = \mathrm{T}$,

$p_{\mathtt{False}} : \mathbb{I} \to X_{\mathtt{bool}}$ with $p_{\mathtt{False}}(\varepsilon) = \mathrm{F}$,

$p_{\mathtt{Zero}} : \mathbb{I} \to X_{\mathtt{nat}}$ with $p_{\mathtt{Zero}}(\varepsilon) = 0$,

$p_{\mathtt{Succ}} : X_{\mathtt{nat}} \to X_{\mathtt{nat}}$ with $p_{\mathtt{Succ}}(n) = n + 1$,

$p_{\underline{n}} : \mathbb{I} \to X_{\mathtt{int}}$ with $p_{\underline{n}}(\varepsilon) = n$ for each $n \in \mathbb{Z}$,

$p_{\mathtt{Pair}} : X_{\mathtt{int}} \times X_{\mathtt{int}} \to X_{\mathtt{pair}}$ with $p_{\mathtt{Pair}}(m, n) = (m, n)$,

$p_{\mathtt{Nil}} : \mathbb{I} \to X_{\mathtt{list}}$ with $p_{\mathtt{Nil}}(\varepsilon) = \varepsilon$,

$p_{\mathtt{Cons}} : X_{\mathtt{int}} \times X_{\mathtt{list}} \to X_{\mathtt{list}}$ with $p_{\mathtt{Cons}}(m, s) = m \triangleleft s$,

where $m \triangleleft s$ is the element of $\mathbb{Z}^*$ obtained by adding m to the beginning of the list s, i.e.,

$$m \triangleleft s = \begin{cases} m\, m_1 \cdots m_n & \text{if } s = m_1 \cdots m_n \text{ with } n \geq 1, \\ m & \text{if } s = \varepsilon. \end{cases}$$

Example 2.2.2 An enumerated signature $\Lambda = (B, K, \Theta)$ with B and K finite can (and in most functional programming languages will) be represented in a form similar to the following, where $b_1, \ldots, b_n$ is some enumeration of the elements in the set B, $k_{j1}, \ldots, k_{jm_j}$ an enumeration of the elements of K_{b_j} for each j and $\Theta(k_{j\ell}) = (\gamma_{j\ell}, b_j)$ for each ℓ, j:

$$\begin{array}{lcl} b_1 & ::= & k_{11}\ \gamma_{11} \mid \cdots \mid k_{1m_1}\ \gamma_{1m_1} \\ b_2 & ::= & k_{21}\ \gamma_{21} \mid \cdots \mid k_{2m_2}\ \gamma_{2m_2} \\ & & \vdots \\ b_n & ::= & k_{n1}\ \gamma_{n1} \mid \cdots \mid k_{nm_n}\ \gamma_{nm_n} \end{array}$$

The enumerated signature $\Lambda = (B, K, \Theta)$ introduced in Example 2.2.1 can thus be represented in the form

$$\begin{array}{l} \texttt{bool} ::= \texttt{True} \mid \texttt{False} \\ \texttt{nat} ::= \texttt{Zero} \mid \texttt{Succ nat} \\ \texttt{int} ::= \cdots\ \texttt{-2} \mid \texttt{-1} \mid \texttt{0} \mid \texttt{1} \mid \texttt{2}\ \cdots \\ \texttt{pair} ::= \texttt{Pair int int} \\ \texttt{list} ::= \texttt{Nil} \mid \texttt{Cons int list} \end{array}$$

Of course, there is a problem here with the type `int`, since $K_{\texttt{int}}$ is infinite, but in all real programming languages this type is 'built-in' and so it does not need to be included in the part of the signature specified by the programmer.

For the remainder of the chapter let $\Lambda = (B, K, \Theta)$ be a signature. For each $b \in B$ put $K_b = \{k \in K : k_\triangleleft = b\}$. The sets K_b, $b \in B$, thus form a partition of the set K. (For instance, in the signature Λ in Example 2.2.1 $K_{\texttt{bool}} = \{\texttt{True}, \texttt{False}\}$, $K_{\texttt{nat}} = \{\texttt{Zero}, \texttt{Succ}\}$, $K_{\texttt{int}} = \underline{\mathit{Int}}$, $K_{\texttt{pair}} = \{\texttt{Pair}\}$ and $K_{\texttt{list}} = \{\texttt{Nil}, \texttt{Cons}\}$.)

The next task is to explain what are the structure-preserving mappings between algebras. Let (X, p) and (Y, q) be Λ-algebras and let $\pi : X \to Y$ be a B-family of mappings, i.e., $\pi_b : X_b \to Y_b$ for each $b \in B$. Then the family π is called a *homomorphism* from (X, p) to (Y, q) if

$$q_k \circ \pi^{k^\triangleright} = \pi_{k_\triangleleft} \circ p_k$$

for each $k \in K$. This fact will also be expressed by saying that $\pi : (X, p) \to (Y, q)$ is a homomorphism.

If Λ is enumerated then $\pi : (X, p) \to (Y, q)$ being a homomorphism means that if $k \in K$ is of type $b_1 \cdots b_n \to b$ then

$$q_k(\pi_{b_1}(x_1), \ldots, \pi_{b_n}(x_n)) = \pi_b(p_k(x_1, \ldots, x_n))$$

must hold for all $(x_1, \ldots, x_n) \in X_{b_1} \times \cdots \times X_{b_n}$, this condition being interpreted as $q_k(\varepsilon) = \pi_b(p_k(\varepsilon))$ when k is of type $\varepsilon \to b$.

Proposition 2.2.1 *(1) The B-family of identity mappings* $\mathrm{id} : X \to X$ *defines a homomorphism from a Λ-algebra (X, p) to itself.*

(2) If $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are homomorphisms then the composition $\varrho \circ \pi$ is a homomorphism from (X, p) to (Z, r).

Proof (1) This follows immediately from Lemma 2.1.3 (1).

(2) Let $k \in K$; then by Lemma 2.1.3 (2)

$$r_k \circ (\varrho \circ \pi)^{k^\triangleright} = r_k \circ \varrho^{k^\triangleright} \circ \pi^{k^\triangleright} = \varrho_{k_\triangleleft} \circ q_k \circ \pi^{k^\triangleright} = \varrho_{k_\triangleleft} \circ \pi_{k_\triangleleft} \circ p_k = (\varrho \circ \pi)_{k_\triangleleft} \circ p_k$$

and hence $\varrho \circ \pi$ is a homomorphism from (X, p) to (Z, r). □

Proposition 2.2.1 implies that there is a category whose objects are Λ-algebras and whose morphisms are homomorphisms between Λ-algebras. With the terminology introduced at the end of Section 2.1, a homomorphism $\pi : (X, p) \to (Y, q)$ is an *isomorphism* if there exists a homomorphism $\varrho : (Y, q) \to (X, p)$ such that $\varrho \circ \pi$ and $\pi \circ \varrho$ are the families of identity mappings (on X and Y respectively). If this is the case then for each $b \in B$ the mapping $\pi_b : X_b \to Y_b$ must be a bijection and ϱ_b must be the set-theoretic inverse π_b^{-1} of π_b (and so in particular ϱ is uniquely determined by π). The converse also holds:

Proposition 2.2.2 *If $\pi : (X, p) \to (Y, q)$ is a homomorphism such that for each $b \in B$ the mapping $\pi_b : X_b \to Y_b$ is a bijection then π is an isomorphism.*

Proof This amounts to showing that the family π^{-1} of set-theoretic inverses is also a homomorphism. Let $k \in K$. Then $q_k \circ \pi^{k^\triangleright} = \pi_{k_\triangleleft} \circ p_k$, and therefore by Lemma 2.1.3 (3) it follows that $p_k \circ (\pi^{-1})^{k^\triangleright} = p_k \circ (\pi^{k^\triangleright})^{-1} = \pi_{k_\triangleleft}^{-1} \circ q_k$, which implies that π^{-1} is a homomorphism. □

We are going to divide up the signatures into two kinds. The set

$$B \setminus \Im(\Theta_\triangleleft) = \{b \in B : K_b = \varnothing\}$$

will be called the *parameter set of* Λ, and is always denoted in what follows by A. Thus A is a proper subset of B, since B and K are both non-empty. The signature Λ will be called *closed* if $A = \varnothing$ (which is the case for the signature in Example 2.2.1) and *open* if $A \neq \varnothing$.

Open signatures are typically involved when dealing with what goes under the name of polymorphism. Consider the signature in Example 2.2.1. This contains a

type `list` with associated operator names `Nil` of type $\varepsilon \to$ `list` and `Cons` of type `int list` $\to$ `list`. In the Λ-algebra (X, p) defined in the example $X_{\mathtt{list}} = \mathbb{Z}^*$ with $p_{\mathtt{Nil}} : \mathbb{I} \to \mathbb{Z}^*$ given by $p_{\mathtt{Nil}}(\varepsilon) = \varepsilon$ and $p_{\mathtt{Cons}} : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Z}^*$ given by $p_{\mathtt{Cons}}(m, s) = m \triangleleft s$. Now if lists of some other type t need to be implemented then the signature would have to be extended by adding a new type, say `tlist`, together with two new operator names, one of type $\varepsilon \to$ `tlist` and the other of type t `tlist` $\to$ `tlist`. Moreover, the signature has to be extended in essentially the same way with a new list type and two new operator names for each type for which lists are required. This is clearly not very satisfactory. What is needed is the possibility of defining once and for all lists of an arbitrary type. The reader should look at Example 2.2.3 on the next page but one in order to get some inkling of how this might be done using an open signature. This topic (i.e., polymorphism) will be dealt with systematically in Chapter 6.

The definitions and results in this chapter do not distinguish explicitly between open and closed signatures. However, many of the results are only really relevant when applied to the 'right' kind of signature.

The following simple construction will play an important role in Chapter 6: Let F be a non-empty set and for each $i \in F$ let $\Lambda_i = (B_i, K_i, \Theta_i)$ be a signature. Then Λ is said to be the *disjoint union* of the signatures Λ_i, $i \in F$, if the following conditions hold:

(1) $B_i \cap B_i = \varnothing$ and $K_i \cap K_i = \varnothing$ whenever $i \neq j$.

(2) $B = \bigcup_{i \in F} B_i$ and $K = \bigcup_{i \in F} K_i$.

(3) $\Theta_i(k) = \Theta(k)$ for all $k \in K_i$, $i \in F$.

In particular, if A_i is the parameter set of Λ_i for each $i \in F$ then $A = \bigcup_{i \in F} A_i$ is the parameter set of Λ. If Λ is the disjoint union of the signatures Λ_i, $i \in F$, and (X^i, p^i) is a Λ_i-algebra for each $i \in F$ then a Λ-algebra (X, p) can be defined by putting $X_b = X^i_b$ for each $b \in B_i$ and $p_k = p^i_k$ for each $k \in K_i$. (X, p) will be called the *sum* of the Λ_i-algebras (X^i, p^i), $i \in F$, and will be denoted by $\oplus_{i \in F}(X^i, p^i)$. The converse also holds:

Lemma 2.2.1 *Suppose Λ is the disjoint union of the signatures Λ_i, $i \in F$. Let (X, p) be a Λ-algebra and for each $i \in F$ put $X^i = X_{|B_i}$ and $p^i = p_{|K_i}$. Then (X^i, p^i) is a Λ_i-algebra and $(X, p) = \oplus_{i \in F}(X^i, p^i)$.*

Proof Straightforward. $\square$

Lemma 2.2.2 *Suppose Λ is the disjoint union of the signatures Λ_i, $i \in F$, and for each $i \in F$ let (X^i, p^i), (Y^i, q^i) be Λ_i-algebras and $\pi^i : (X^i, p^i) \to (Y^i, q^i)$ be a homomorphism. Then $\pi : \oplus_{i \in F}(X^i, p^i) \to \oplus_{i \in F}(Y^i, q^i)$ is a homomorphism, where π is given by $\pi_b = \pi^i_b$ for each $b \in B_i$.*

Proof Straightforward. □

A signature $\Lambda' = (B', K', \Theta')$ is said to be an *extension* of Λ if $B \subset B'$, $K \subset K'$ with $\Theta = \Theta'_{|K}$. Thus if $k \in K$ then $k^\triangleright$ and $k_\triangleleft$ have the same meaning in both Λ and Λ'. Note that Λ is trivially an extension of itself.

Let Λ' be an extension of Λ. A Λ'-algebra (Y, q) is then called an *extension* of a Λ-algebra (X, p) if $X_b \subset Y_b$ for each $b \in B$ and p_k is the restriction of q_k to $X^{k^\triangleright}$ for each $k \in K$. (The case $\Lambda' = \Lambda$ is certainly not excluded here, and in fact it will occur frequently in what follows.) Note that if (Y, q) is a Λ'-algebra then $(Y_{|B}, q_{|K})$ is a Λ-algebra. Moreover, if π is a homomorphism from (Y, q) to (Z, r) then the family $\pi_{|B}$ is a homomorphism from $(Y_{|B}, q_{|K})$ to $(Z_{|B}, r_{|K})$.

Let us now look at the special case of a *single-sorted* signature, i.e., a signature of the form $(\mathbb{I}, K, \Theta)$. In this case there is no choice for the second component of Θ (since there is only one mapping possible from K to $\mathbb{I}$) and so a single-sorted signature can be regarded as being a pair (K, ϑ) consisting of a set K and a mapping $\vartheta : K \to \mathsf{FSets}$, recalling that $\mathcal{T}(\mathbb{I})$ can be identified with the class FSets of all finite sets.

Let $\Lambda = (K, \vartheta)$ be a single-sorted signature; then (identifying a $\mathbb{I}$-family Z with the object Z_ε) a Λ-algebra is here a pair (X, p) consisting of a set X and a K-family of mappings p with $p_k : X^{\vartheta(k)} \to X$ for each $k \in K$.

Consider now the even more special case of an enumerated single-sorted signature $\Lambda = (K, \vartheta)$. Then, since $\mathbb{I}^*$ can clearly be identified with the set of natural numbers $\mathbb{N}$, ϑ can here be regarded as a mapping from K to $\mathbb{N}$. If (X, p) is a Λ-algebra then p_k is a mapping from the cartesian product $X^{\vartheta(k)}$ to X, so $\vartheta(k)$ is just the number of arguments taken by the operator p_k.

If $\Lambda = (B, K, \Theta)$ is an arbitrary signature then there is a single-sorted signature $\Lambda^o = (K, \vartheta)$ with $\vartheta : K \to \mathsf{FSets}$ defined by $\vartheta(k) = \langle k^\triangleright \rangle = \mathrm{dom}(k^\triangleright)$ for each $k \in K$. This means that Λ^o is obtained from Λ by no longer distinguishing between the various types. Now let (Y, p) be a Λ^o-algebra, so Y is a set and $p_k : X^{\vartheta(k)} \to X$ for each $k \in K$; for each $b \in B$ put $X_b = Y$. Then $X^{k^\triangleright} = Y^{\vartheta(k)}$ for each $k \in K$, since any mapping from $\vartheta(k) = \langle k^\triangleright \rangle$ to $\bigcup_{\eta \in \langle k^\triangleright \rangle} X_{k^\triangleright \eta} = Y$ is automatically typed, which implies that (X, p) is a Λ-algebra. This almost trivial method of obtaining Λ-algebras turns out to be surprisingly useful.

Example 2.2.3 Consider the signature $\Lambda = (B, K, \Theta)$ with

$B = \{\texttt{bool}, \texttt{atom}, \texttt{int}, \texttt{pair}, \texttt{list}, \texttt{lp}, \texttt{x}, \texttt{y}, \texttt{z}\}$,

$K = \{\texttt{True}, \texttt{False}, \texttt{Atom}, \texttt{Pair}, \texttt{Nil}, \texttt{Cons}, \texttt{L}, \texttt{P}\} \cup \underline{Int}$,

and with $\Theta : K \to B^* \times B$ defined by

$\Theta(\texttt{True}) = \Theta(\texttt{False}) = (\varepsilon, \texttt{bool})$,

$\Theta(\texttt{Atom}) = (\varepsilon, \texttt{atom})$,

$\Theta(\texttt{Pair}) = (\texttt{x}\ \texttt{y}, \texttt{pair})$,

$\Theta(\texttt{Nil}) = (\varepsilon, \texttt{list})$, $\Theta(\texttt{Cons}) = (\texttt{z}\ \texttt{list}, \texttt{list})$,

$\Theta(\texttt{L}) = (\texttt{list}, \texttt{lp})$, $\Theta(\texttt{P}) = (\texttt{pair}, \texttt{lp})$,

$\Theta(\underline{n}) = (\varepsilon, \texttt{int})$ for each $n \in \mathbb{Z}$.

The parameter set is here the set $A = \{\texttt{x}, \texttt{y}, \texttt{z}\}$. Using the conventions introduced in Example 2.2.2, this signature can be represented as

$$
\begin{aligned}
\texttt{bool} &::= \texttt{True} \mid \texttt{False}\\
\texttt{atom} &::= \texttt{Atom}\\
\texttt{int} &::= \cdots\ {-2} \mid -1 \mid 0 \mid 1 \mid 2\ \cdots\\
\texttt{pair} &::= \texttt{Pair x y}\\
\texttt{list} &::= \texttt{Nil} \mid \texttt{Cons z list}\\
\texttt{lp} &::= \texttt{L list} \mid \texttt{P pair}
\end{aligned}
$$

Let V be an A-family of sets. A Λ-algebra (X, p) with $X_{|A} = V$ can then be defined by putting

$X_{\texttt{bool}} = \mathbb{B}$, $X_{\texttt{atom}} = \mathbb{I}$, $X_{\texttt{int}} = \mathbb{Z}$, $X_{\texttt{pair}} = V_{\texttt{x}} \times V_{\texttt{y}}$, $X_{\texttt{list}} = V_{\texttt{z}}^*$,

$X_{\texttt{lp}} = (V_{\texttt{x}} \times V_{\texttt{y}}) \cup V_{\texttt{z}}^*$ (this union being considered to be disjoint),

$X_{\texttt{x}} = V_{\texttt{x}}$, $X_{\texttt{y}} = V_{\texttt{y}}$, $X_{\texttt{z}} = V_{\texttt{z}}$,

$p_{\texttt{True}} : \mathbb{I} \to X_{\texttt{bool}}$ with $p_{\texttt{True}}(\varepsilon) = \mathrm{T}$,

$p_{\texttt{False}} : \mathbb{I} \to X_{\texttt{bool}}$ with $p_{\texttt{False}}(\varepsilon) = \mathrm{F}$,

$p_{\texttt{Atom}} : \mathbb{I} \to X_{\texttt{atom}}$ with $p_{\texttt{Atom}}(\varepsilon) = \varepsilon$,

$p_{\underline{n}} : \mathbb{I} \to \mathbb{Z}$ with $p_{\underline{n}}(\varepsilon) = n$ for each $n \in \mathbb{Z}$,

$p_{\texttt{Pair}} : V_{\texttt{x}} \times V_{\texttt{y}} \to X_{\texttt{pair}}$ with $p_{\texttt{Pair}}(x, y) = (x, y)$,

$p_{\texttt{Nil}} : \mathbb{I} \to X_{\texttt{list}}$ with $p_{\texttt{Nil}}(\varepsilon) = \varepsilon$,

$p_{\texttt{Cons}} : V_{\texttt{z}} \times X_{\texttt{list}} \to X_{\texttt{list}}$ with $p_{\texttt{Cons}}(z, s) = m \triangleleft s$,

$p_{\texttt{L}} : X_{\texttt{list}} \to X_{\texttt{lp}}$ with $p_{\texttt{L}}(s) = s$,

$p_{\texttt{P}} : X_{\texttt{pair}} \to X_{\texttt{lp}}$ with $p_{\texttt{P}}(p) = p$.

In Chapter 6 an augmented form of the representation of the signature Λ will be introduced, providing the types `pair`, `list` and `lp` with the parameters from the set A on which they depend.

2.3 Invariant families and minimal algebras

For the whole of the section let (X, p) be a Λ-algebra.

Let $Y \subset X$ (i.e., Y is a B-family of sets with $Y_b \subset X_b$ for each $b \in B$). The family Y is said to be *invariant in* (X, p), or just *invariant*, if $p_k(Y^{k^\triangleright}) \subset Y_{k_\triangleleft}$ for all $k \in K$. In particular, the family X is itself trivially invariant.

A related notion is that of a subalgebra: A Λ-algebra (Y, q) is a *subalgebra* of (X, p) if $Y \subset X$ and q_k is the restriction of p_k to $Y^{k^\triangleright}$ for each $k \in K$. In this case the family Y is clearly invariant. Conversely, let Y be any invariant family and for each $k \in K$ let q_k denote the restriction of p_k to $Y^{k^\triangleright}$, so q_k can be considered as a mapping from $Y^{k^\triangleright}$ to $Y_{k_\triangleleft}$. Then (Y, q) is a subalgebra of (X, p). This means there is a one-to-one correspondence between invariant families and subalgebras of (X, p). If Y is an invariant family then the corresponding subalgebra (Y, q) is called the *subalgebra associated with* Y.

Lemma 2.3.1 *Let* $\pi : (X, p) \to (Y, q)$ *be a homomorphism.*

(1) If the family $\breve{X}$ *is invariant in* (X, p) *then the family* $\breve{Y}$ *given by* $\breve{Y}_b = \pi_b(\breve{X}_b)$ *for each* $b \in B$ *is invariant in* (Y, q)*.*

(2) If the family $\breve{Y}$ *is invariant in* (Y, q) *then the family* $\breve{X}$ *given by* $\breve{X}_b = \pi_b^{-1}(\breve{Y}_b)$ *for each* $b \in B$ *is invariant in* (X, p)*.*

Proof (1) Let $k \in K$; then by Lemma 2.1.3 (3) $\pi^{k^\triangleright}(\breve{X}^{k^\triangleright}) = \breve{Y}^{k^\triangleright}$ and hence

$$q_k(\breve{Y}^{k^\triangleright}) = q_k(\pi^{k^\triangleright}(\breve{X}^{k^\triangleright})) = \pi_{k_\triangleleft}(p_k(\breve{X}^{k^\triangleright})) \subset \pi_{k_\triangleleft}(\breve{X}_{k_\triangleleft}) = \breve{Y}_{k_\triangleleft} .$$

The family $\breve{Y}$ is thus invariant in (Y, q).

(2) Let $k \in K$; then by Lemma 2.1.3 (3) $\pi^{k^\triangleright}(\breve{X}^{k^\triangleright}) \subset \breve{Y}^{k^\triangleright}$, since $\pi_b(\breve{X}_b) \subset \breve{Y}_b$ for each $b \in B$. Hence

$$\pi_{k_\triangleleft}(p_k(\breve{X}^{k^\triangleright})) = q_k(\pi^{k^\triangleright}(\breve{X}^{k^\triangleright})) \subset q_k(\breve{Y}^{k^\triangleright}) \subset \breve{Y}_{k_\triangleleft}$$

and therefore $p_k(\breve{X}^{k^\triangleright}) \subset \pi_{k_\triangleleft}^{-1}(\breve{Y}_{k_\triangleleft}) = \breve{X}_{k_\triangleleft}$. This implies that the family $\breve{X}$ is invariant in (X, p). $\square$

Lemma 2.3.1 says that both the image and the pre-image of a subalgebra under a homomorphism are again subalgebras. In what follows let U be a B-family of sets with $U \subset X$.

Lemma 2.3.2 *If* Y *is an invariant family in* (X, p) *containing* U *and*

$$\breve{Y}_b = U_b \cup \bigcup_{k \in K_b} p_k(Y^{k^\triangleright})$$

for each $b \in B$ *then the family* $\breve{Y}$ *is invariant in* (X, p) *and* $U \subset \breve{Y} \subset Y$*.*

Proof By definition $U \subset \breve{Y}$, and $\breve{Y} \subset Y$ holds because Y is invariant. Moreover, $\breve{Y}$ is invariant, since then $p_k(\breve{Y}^{k^\triangleright}) \subset p_k(Y^{k^\triangleright}) \subset \breve{Y}_{k_\triangleleft}$ for all $k \in K$. □

Lemma 2.3.3 *There is a minimal invariant family containing* U *(i.e., there is an invariant family* $\hat{X}$ *with* $U \subset \hat{X}$ *such that if* Y *is any invariant family containing* U *then* $\hat{X} \subset Y$*).*

Proof As already noted, the family X is itself invariant, and it contains of course U. Moreover, it is easy to see that an arbitrary intersection of invariant families is again invariant. (More precisely, if X^t is an invariant family for each $t \in T$ and $\grave{X}_b = \bigcap_{t \in T} X^t_b$ for each $b \in B$ then $\grave{X}$ is also invariant.) The intersection of all the invariant families containing U is thus the required minimal family. In fact, this minimal family $\hat{X}$ can be given somewhat more explicitly: For each $n \in \mathbb{N}$ define a family $\hat{X}^n \subset X$ by putting $\hat{X}^0 = U$ and for each $n \in \mathbb{N}$, $b \in B$ letting $\hat{X}^{n+1}_b = \hat{X}^n_b \cup \bigcup_{k \in K_b} \Im(\hat{p}^n_k)$, where $\hat{p}^n_k$ is the restriction of p_k to $(\hat{X}^n)^{k^\triangleright}$. Then it is straightforward to check that $\hat{X}_b = \bigcup_{n \in \mathbb{N}} \hat{X}^n_b$ for each $b \in B$. This shows that $\hat{X}_b$ consists exactly of those elements of X_b which can be 'constructed' in a finite number of steps out of elements from the family U and elements which have already been 'constructed'. □

Lemma 2.3.4 *If* $\hat{X}$ *is the minimal invariant family containing* U *then*

$$\hat{X}_b = U_b \cup \bigcup_{k \in K_b} p_k(\hat{X}^{k^\triangleright})$$

for each $b \in B$.

Proof This follows immediately from Lemma 2.3.2. □

If $\hat{X}$ is the minimal invariant family containing U then the associated subalgebra will be referred to as the *minimal subalgebra of* (X, p) *containing* U.

The Λ-algebra (X, p) is now said to be U*-minimal* if X is the only invariant family containing U. Note that the minimal subalgebra of (X, p) containing U is always a U-minimal Λ-algebra.

Proposition 2.3.1 *If* (X, p) *is* U*-minimal then for each* $b \in B$

$$U_b \cup \bigcup_{k \in K_b} \Im(p_k) = X_b\ .$$

Proof This follows immediately from Lemma 2.3.4. □

Consider the special case when $U = \varnothing$: The minimal invariant family containing $\varnothing$ is of course just the minimal invariant family, and the associated subalgebra will be referred to as the *minimal subalgebra of* (X, p). Moreover, (X, p) is said to be *minimal* if X is the only invariant family, thus (X, p) is minimal if and only if it is $\varnothing$-minimal.

If (X, p) is minimal then by Proposition 2.3.1 $\bigcup_{k \in K_b} \Im(p_k) = X_b$ for each $b \in B$.

As in Section 2.2 let A be the parameter set of Λ (so $A = \{b \in B : K_b = \varnothing\}$). Note that if (X, p) is U-minimal then by Proposition 2.3.1 $X_{|A} = U_{|A}$. In particular, if (X, p) is minimal then $X_{|A} = \varnothing$. This indicates that minimality is usually not an appropriate requirement when the signature is open (i.e., when $A \neq \varnothing$).

The converse of Proposition 2.3.1 does not hold in general. However, the condition occurring there can be combined with a second condition to give a useful sufficient criterion for being minimal. A B-family of mappings $\#$ with $\#_b : X_b \to \mathbb{N}$ for each $b \in B$ will be called a *grading* for (X, p) if for each $k \in K$

$$\#_{k^\triangleright \eta}(v(\eta)) < \#_{k_\triangleleft}(p_k(v))$$

for all $\eta \in \langle k^\triangleright \rangle$. If there exists a grading then (X, p) is said to be *graded*. If Λ is enumerated then a family $\#$ being a grading means that

$$\#_{b_j}(x_j) < \#_b(p_k(x_1, \ldots, x_n))$$

must hold for each $j = 1, \ldots, n$ for each $(x_1, \ldots, x_n) \in X_{b_1} \times \cdots \times X_{b_n}$ whenever $k \in K$ is of type $b_1 \cdots b_n \to b$.

Lemma 2.3.5 *If (X, p) is graded and for each $b \in B$*

$$U_b \cup \bigcup_{k \in K_b} \Im(p_k) = X_b$$

then X is U-minimal.

Proof Let $\#$ be a grading for (X, p), let $\hat{X}$ be the minimal invariant family containing U and suppose $\hat{X} \neq X$. There thus exists $b \in B$ and $x \in X_b \setminus \hat{X}_b$ such that $\#_b(x) \leq \#_{b'}(x')$ whenever $x' \in X_{b'} \setminus \hat{X}_{b'}$ for some $b' \in B$. Then $x \in \Im(p_k)$ some $k \in K_b$, since $U_b \subset \hat{X}_b$, and so there exists $v \in X^{k^\triangleright}$ with $x = p_k(v)$. But it then follows that $\#_{k^\triangleright \eta}(v(\eta)) < \#_{k_\triangleleft}(x)$ and hence that $v(\eta) \in \hat{X}_{k^\triangleright \eta}$ for each $\eta \in \langle k^\triangleright \rangle$ (by the minimality of $\#_b(x)$). However, this implies $x \in \hat{X}_b$, since the family $\hat{X}$ is invariant, which is a contradiction. □

Proposition 2.3.2 *If (X, p) is graded then it is U-minimal if and only if*

$$U_b \cup \bigcup_{k \in K_b} \Im(p_k) = X_b$$

for each $b \in B$.

Proof This follows immediately from Proposition 2.3.1 and Lemma 2.3.5. □

There is an obvious grading for the Λ-algebra (X, p) defined in Example 2.2.1, and it is thus easy to check that this algebra is minimal.

Proposition 2.3.3 *(1) If (X, p) is U-minimal and π and ϱ are homomorphisms from (X, p) to a Λ-algebra (Y, q) with $\pi_b(x) = \varrho_b(x)$ for all $x \in U_b$, $b \in B$, then $\pi = \varrho$. In particular, if (X, p) is minimal then there exists at most one homomorphism from (X, p) to a Λ-algebra (Y, q).*

(2) If (X, p) is U-minimal then any homomorphism π from a Λ-algebra (Y, q) to (X, p) with $U_b \subset \pi_b(Y_b)$ for each $b \in B$ is surjective (i.e., π_b is surjective for each $b \in B$). In particular, if (X, p) is minimal then any homomorphism $\pi : (Y, q) \to (X, p)$ is surjective.

Proof (1) For each $b \in B$ let $\breve{X}_b = \{x \in X_b : \pi_b(x) = \varrho_b(x)\}$; thus $U \subset \breve{X}$. Consider $k \in K$ and let $v \in \breve{X}^{k^\triangleright}$; then $v(\eta) \in \breve{X}_{k^\triangleright \eta}$ for each $\eta \in \langle k^\triangleright \rangle$ and hence

$$\pi^{k^\triangleright}(v)(\eta) = \pi_{k^\triangleright \eta}(v(\eta)) = \varrho_{k^\triangleright \eta}(v(\eta)) = \varrho^{k^\triangleright}(v)(\eta)$$

which implies that $\pi^{k^\triangleright}(v) = \varrho^{k^\triangleright}(v)$. Therefore

$$\pi_{k_\triangleleft}(p_k(v)) = q_k(\pi^{k^\triangleright}(v)) = q_k(\varrho^{k^\triangleright}(v)) = \varrho_{k_\triangleleft}(p_k(v)) \ ,$$

i.e., $p_k(v) \in \breve{X}_{k_\triangleleft}$. This shows that the family $\breve{X}$ is invariant and thus $\breve{X} = X$, since (X, p) is U-minimal. In other words, $\pi = \varrho$.

(2) This follows immediately from Lemma 2.3.1 (1). □

2.4 Initial algebras

Using the terminology introduced in Section 2.1, a Λ-algebra (X, p) is said to be *initial* if for each Λ-algebra (Y, q) there exists a unique homomorphism from (X, p) to (Y, q). Let us note here that for open signatures there is a more appropriate notion which will be dealt with in Section 2.6.

Initial algebras will be characterised as those that are minimal and possess a further property, here called regularity. The existence of an initial Λ-algebra will be established by showing that a minimal regular Λ-algebra exists.

Again let $A = \{b \in B : K_b = \varnothing\}$ be the parameter set of Λ. A Λ-algebra (X, p) is said to be *regular* if for each $b \in B \setminus A$ and each $x \in X_b$ there exists a unique $k \in K_b$ and a unique element $v \in X^{k^\triangleright}$ such that $p_k(v) = x$. Thus (X, p) is regular if and only if the mapping p_k is injective for each $k \in K$ and for each $b \in B \setminus A$ the sets $\Im(p_k)$, $k \in K_b$, form a partition of X_b.

In particular, the Λ-algebra (X, p) in Example 2.2.1 is clearly regular.

It is useful to introduce a further condition, which turns out to be equivalent to being regular for minimal algebras. A Λ-algebra (X, p) is said to be *unambiguous* if the mapping p_k is injective for each $k \in K$ and for each $b \in B$ the sets $\Im(p_k)$, $k \in K_b$, are disjoint subsets of X_b. (Of course, in the definition of being unambiguous it would make no difference if B were replaced by $B \setminus A$, since $K_a = \varnothing$ for each $a \in A$.)

Proposition 2.4.1 *There exists an initial Λ-algebra. Moreover, the following are equivalent for a Λ-algebra (X, p):*

(1) (X, p) is initial.
(2) (X, p) is minimal and regular.
(3) (X, p) is minimal and unambiguous.

Proof This occupies the rest of the section. $\square$

The characterisation of initial algebras in Proposition 2.4.1 is sometimes referred to as stating that initial algebras are exactly those for which there is *no junk* and *no confusion* (i.e., those which are minimal and unambiguous).

Remark: Recall that if (X, p) is minimal then by Proposition 2.3.2 $X_{|A} = \varnothing$, and so Proposition 2.2.1 implies that also $X_{|A} = \varnothing$ for each initial Λ-algebra (X, p). (This indicates why being initial is usually not an appropriate requirement when the signature is open.) Note further that this property of minimal algebras means that, as far as Proposition 2.4.1 is concerned, $B \setminus A$ could be replaced by B in the definition of being regular. However, many of the Λ-algebras which will be

considered are not minimal and then the definition of regularity made above is the 'correct' one.

Before going any further consider again the natural number algebras discussed in the Introduction. The equivalence of initial natural number algebras and Peano triples is easily seen to be just a special case of Proposition 2.4.1.

The equivalence of (2) and (3) in Proposition 2.4.1 is dealt with by the following simple lemma:

Lemma 2.4.1 *A minimal* Λ*-algebra is unambiguous if and only if it is regular.*

Proof It is clear that any regular Λ-algebra is unambiguous. Conversely, an unambiguous Λ-algebra (X, p) will be regular if $\bigcup_{k \in K_b} \Im(p_k) = X_b$ for each $b \in B \setminus A$, and this holds for minimal Λ-algebras by Proposition 2.3.1. $\square$

The main steps in the proof of Proposition 2.4.1 are first to show that there exists a minimal regular Λ-algebra and then to show that any such Λ-algebra is initial.

In order to get started an unambiguous Λ-algebra is needed, and the following trivial observation is useful here: Let $\Lambda^o = (K, \vartheta)$ be the single-sorted signature defined at the end of Section 2.2 (with the mapping $\vartheta : K \to \mathsf{FSets}$ given by $\vartheta(k) = \langle k^{\triangleright} \rangle$ for each $k \in K$, recalling that $\langle k^{\triangleright} \rangle = \mathrm{dom}(k^{\triangleright})$). Let (Y, p) be a Λ^o-algebra and (X, p) be the Λ-algebra with $X_b = Y$ for each $b \in B$.

Lemma 2.4.2 *If the* Λ^o*-algebra* (Y, p) *is unambiguous then so is the* Λ*-algebra* (X, p).

Proof This is clear. $\square$

Lemma 2.4.3 *There exists an unambiguous* Λ*-algebra.*

Proof By Lemma 2.4.2 it is enough to show that if $\Lambda' = (K, \vartheta)$ is a single-sorted signature then there exists a unambiguous Λ'-algebra. The construction given below is just one of many possibilities of defining such an algebra.

Let $M = M_o \cup K$, where M_o is the set of all pairs of the form (k, η) with $k \in K$ and $\eta \in \vartheta(k)$, and let X be the set of all non-empty finite subsets of M^*. Now if $k \in K$ with $\vartheta(k) = \varnothing$ then define $p_k : \mathbb{I} \to X$ by letting $p_k(\varepsilon) = \{k\}$ (where here k is the list consisting of the single component k), and if $k \in K$ with $\vartheta(k) \neq \varnothing$ then define a mapping $p_k : X^{\vartheta(k)} \to X$ by letting

$$p_k(v) = \{k\} \cup \bigcup_{\eta \in \vartheta(k)} \{(k, \eta) \triangleleft s : s \in v(\eta)\}$$

for each $v \in X^{\vartheta(k)}$, where $\triangleleft : M \times M^* \to M^*$ is the (infix) operation of adding an element to the beginning of a list. This gives a Λ'-algebra (X, p). But it is easily checked that $\Im(p_{k_1})$ and $\Im(p_{k_2})$ are disjoint subsets of X whenever $k_1 \neq k_2$, and also that p_k is injective for each $k \in K$. Hence (X, p) is unambiguous. $\square$

Lemma 2.4.4 *There exists a minimal regular Λ-algebra.*

Proof By Lemma 2.4.3 there exists an unambiguous Λ-algebra (X, p). But then any subalgebra of (X, p) is also unambiguous. In particular, the minimal subalgebra is minimal and unambiguous, and hence by Lemma 2.4.1 it is minimal and regular. $\square$

Lemma 2.4.5 *Let (X, p) be a minimal regular Λ-algebra. Then there exists a unique B-family $\#$ with $\#_b : X_b \to \mathbb{N}$ for each $b \in B$ with $\#_{k_\triangleleft}(p_k(\varepsilon)) = 0$ if $k \in K$ with $k^\triangleright = \varepsilon$ and such that*

$$\#_{k_\triangleleft}(p_k(v)) = 1 + \max\{\#_{k^\triangleright\eta}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\}$$

for all $v \in X^{k^\triangleright}$ whenever $k \in K$ with $k^\triangleright \neq \varepsilon$.

Proof Let us emphasise again that $X_{|A} = \varnothing$, since (X, p) is minimal. The family $\#$ will be obtained as the limit of a sequence $\{\#^m\}_{m \geq 0}$, where $\#^m$ is a B-family of mappings with $\#^m_b : X_b \to \mathbb{N}$ for each $b \in B$. First define $\#^0_b = 0$ for each $b \in B$. Next suppose that the family $\#^m$ has already been defined for some $m \in \mathbb{N}$. Then, since (X, p) is regular and $X_{|A} = \varnothing$, there is a unique family of mappings $\#^{m+1}$ such that $\#^{m+1}_{k_\triangleleft}(p_k(\varepsilon)) = 0$ if $k \in K$ with $k^\triangleright = \varepsilon$ and such that

$$\#^{m+1}_{k_\triangleleft}(p_k(v)) = 1 + \max\{\#^m_{k^\triangleright\eta}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\}$$

for all $v \in X^{k^\triangleright}$ whenever $k \in K$ with $k^\triangleright \neq \varepsilon$. By induction this defines the family $\#^m$ for each $m \in \mathbb{N}$.

Now $\#^m \leq \#^{m+1}$ holds for each $m \in \mathbb{N}$ (i.e., $\#^m_b(x) \leq \#^{m+1}_b(x)$ for all $x \in X_b$, $b \in B$): This follows by induction on m, since $\#^0 \leq \#^1$ holds by definition and if $\#^m \leq \#^{m+1}$ for some $m \in \mathbb{N}$ and $k \in K$ with $k^\triangleright \neq \varepsilon$ then

$$\begin{aligned} \#^{m+1}_{k_\triangleleft}(p_k(v)) &= 1 + \max\{\#^m_{k^\triangleright\eta}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\} \\ &\leq 1 + \max\{\#^{m+1}_{k^\triangleright\eta}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\} = \#^{m+2}_{k_\triangleleft}(p_k(v)) \end{aligned}$$

for all $v \in X^{k^\triangleright}$, which implies that $\#^{m+1} \leq \#^{m+2}$. Moreover, the sequence $\{\#^m_b(x)\}_{m \geq 0}$ is bounded for each $x \in X_b$, $b \in B$: Let X'_b denote the set of those elements $x \in X_b$ for which this is the case. Then it is easily checked that the B-family X' is invariant, and hence $X' = X$, since (X, p) is minimal.

Let $x \in X_b$; then by the above $\{\#_b^m(x)\}_{m\geq 0}$ is a bounded increasing sequence from $\mathbb{N}$, and so there exists an element $\#_b(x) \in \mathbb{N}$ such that $\#_b^m(x) = \#_b(x)$ for all but finitely many m. This defines a mapping $\#_b : X_b \to \mathbb{N}$ for each $b \in B$, and it immediately follows that the family $\#$ has the required property. It remains to show the uniqueness, so suppose $\#'$ is another B-family of mappings with this property. For each $b \in B$ let $X'_b = \{x \in X_b : \#'_b(x) = \#_b(x)\}$; then the family X' is clearly invariant and hence $X' = X$, since (X, p) is minimal. □

It can now be shown that a minimal regular Λ-algebra (X, p) is initial: Let (Y, q) be any Λ-algebra; then it is enough to just construct a homomorphism $\pi : (X, p) \to (Y, q)$, since Proposition 2.3.1 (1) implies that this homomorphism is unique.

Let $\#$ be the family of mappings given by Lemma 2.4.5 (with $\#_b : X_b \to \mathbb{N}$ for each $b \in B$) and for each $b \in B$, $m \in \mathbb{N}$ let $X_b^m = \{x \in X_b : \#_b(x) = m\}$. Define π by defining π_b on X_b^m for each $b \in B$ using induction on m. Let $x \in X_b^0$; then, since (X, p) is regular and $X_{|A} = \varnothing$, there exists a unique $k \in K_b$ and a unique element $v \in X^{k^\triangleright}$ such that $p_k(v) = x$, and here $k^\triangleright = \varepsilon$ and so $x = p_k(\varepsilon)$. Thus put $\pi_b(x) = q_k(\varepsilon)$, which defines π_b on X_b^0 for each $b \in B$. Now let $m > 0$ and suppose $\pi_{b'}$ has already been defined on $X_{b'}^n$ for all $n < m$ and all $b' \in B$. Let $x \in X_b^m$; again using regularity and the fact that $X_{|A} = \varnothing$, there exists a unique $k \in K_b$ and a unique $v \in X^{k^\triangleright}$ such that $x = p_k(v)$. In this case $v \in X^{k^\triangleright}$ with $k^\triangleright \neq \varepsilon$, and by the defining property of $\#$ it follows that $\#_{k^\triangleright\eta}(v(\eta)) < m$ for each $\eta \in \langle k^\triangleright \rangle$, which means $\pi_{k^\triangleright\eta}(v(\eta))$ is already defined for each $\eta \in \langle k^\triangleright \rangle$ and hence that $\pi^{k^\triangleright}(v)$ is already defined (i.e., $\pi^{k^\triangleright}(v)$ is the element v' of $Y^{k^\triangleright}$ given by $v'(\eta) = \pi_{k^\triangleright\eta}(v(\eta))$ for each $\eta \in \langle k^\triangleright \rangle$). Thus here put $\pi_b(x) = q_k(\pi^{k^\triangleright}(v))$. In this way π_b is defined on X_b^m for each $b \in B$ and each $m \in \mathbb{N}$, and the family π is a homomorphism more-or-less by construction.

This shows that any minimal regular Λ-algebra is initial which, together with Lemma 2.4.4, also implies that an initial Λ-algebra exists. In order to show that an initial Λ-algebra is minimal and regular the following fact is needed:

Lemma 2.4.6 *A Λ-algebra isomorphic to a minimal unambiguous Λ-algebra is itself minimal and unambiguous. Thus by Lemma 2.4.1 a Λ-algebra isomorphic to a minimal regular Λ-algebra is itself minimal and regular.*

Proof Let (Y, q) a minimal unambiguous Λ-algebra and $\pi : (X, p) \to (Y, q)$ be an isomorphism. If X' is an invariant family in (X, p) then Lemma 2.3.1 (1) implies that $\pi_b(X'_b) = Y_b$ for each $b \in B$, since Y is the only invariant family in (Y, q). Thus $X'_b = \pi_b^{-1}(\pi_b(X'_b)) = \pi_b^{-1}(Y_b) = X_b$ for each $b \in B$, i.e., $X' = X$, and this shows that (X, p) is minimal. Now if $k \in K_b$ and $x \in \Im(p_k)$ then by the definition of a homomorphism $\pi_b(x) \in \Im(q_k)$. It immediately follows that if k_1, $k_2 \in K_b$ with $k_1 \neq k_2$ then $\Im(p_{k_1}) \cap \Im(p_{k_2}) = \varnothing$. Finally, $q_k \circ \pi^{k^\triangleright} = \pi_{k_\triangleleft} \circ p_k$ and, moreover,

q_k is injective, $\pi_{k_\triangleleft}$ is a bijection and by Lemma 2.1.3 (3) $\pi^{k^\triangleright}$ is also a bijection. Thus p_k is injective for each $k \in K$. $\square$

Now consider the property P of Λ-algebras of being minimal and regular. Then it has already been shown that there exists a Λ-algebra having property P and that every Λ-algebra having property P is initial. Moreover, Lemma 2.4.4 shows that any Λ-algebra isomorphic to a Λ-algebra having property P has property P. Thus by Proposition 2.1.2 every initial Λ-algebra has property P, and this completes the proof of Proposition 2.4.1. $\square$

Proposition 2.4.1 implies that the Λ-algebra (X, p) in Example 2.2.1 is initial.

A type $b \in B \setminus A$ will be said to be *primitive* if $k^\triangleright = \varepsilon$ for each $k \in K_b$. (In the signature Λ in Example 2.2.1 the primitive types are therefore `bool` and `int`.) Note that if (X, p) is an initial Λ-algebra and $b \in B \setminus A$ is a primitive type then the mapping $k \mapsto p_k(\varepsilon)$ maps K_b bijectively onto X_b. (For the Λ-algebra (X, p) in Example 2.2.1 this gives the obvious bijections between $K_{\texttt{bool}} = \{\texttt{True}, \texttt{False}\}$ and $X_{\texttt{bool}} = \mathbb{B} = \{\mathrm{T}, \mathrm{F}\}$ and between $K_{\texttt{int}} = \underline{Int}$ and $X_{\texttt{int}} = \mathbb{Z}$.)

Lemma 2.4.7 *The minimal subalgebra of an unambiguous Λ-algebra is initial.*

Proof This follows from Proposition 2.4.1 and Lemma 2.4.1, since, as was already noted, any subalgebra of an unambiguous algebra is unambiguous. $\square$

Let $\Lambda' = (B', K, \Theta')$ be a signature which is an extension of Λ. The following is a useful extension of Proposition 2.4.1:

Proposition 2.4.2 *Let (X, p) be an initial Λ-algebra. Then there exists an initial Λ'-algebra which is an extension of (X, p).*

Proof By Proposition 2.4.1 there exists an initial Λ'-algebra (Z, r) and the pair $(Y, q) = (Z_{|B}, r_{|K})$ is then a Λ-algebra; let $(\hat{Y}, \hat{q})$ be the minimal subalgebra of (Y, q). Now by Proposition 2.4.1 and Lemma 2.4.1 (Z, r) is unambiguous and thus (Y, q) is also unambiguous. Hence by Lemma 2.4.7 $(\hat{Y}, \hat{q})$ is an initial Λ-algebra, and by construction (Z, r) is an extension of $(\hat{Y}, \hat{q})$. The unique isomorphism from (X, p) to $(\hat{Y}, \hat{q})$ can therefore be used to construct an initial Λ'-algebra which is an extension of (X, p). $\square$

We end the section by noting that if $\Lambda = (K, \vartheta)$ is a single-sorted signature then the unique homomorphism from an initial Λ-algebra to a Λ-algebra (Y, q) exists even when Y is a class. This observation forms the basis for several of the constructions which will occur in later chapters.

If $\mathcal{C}$ is a class and J a set then $\mathcal{C}^J$ will be used to denote the class of all mappings from J to $\mathcal{C}$; in particular, $\mathcal{C}^\varnothing = \mathbb{I}$. If X is a set and $\pi : X \to \mathcal{C}$ is a mapping then π^J will be used to denote the mapping from X^J to $\mathcal{C}^J$ defined by

$$\pi^J(u)(\eta) = \pi(u(\eta))$$

for all $u \in X^J$, $\eta \in J$.

Let $\Lambda = (K, \vartheta)$ be a single-sorted signature, so $\vartheta : K \to \mathsf{FSets}$ and a Λ-algebra is here a pair (X, p) consisting of a set X and a K-family of mappings p with $p_k : X^{\vartheta(k)} \to X$ for each $k \in K$. Let $\mathcal{C}$ be a class and let φ be a K-family of mappings with $\varphi_k : \mathcal{C}^{\vartheta(k)} \to \mathcal{C}$ for each $k \in K$. Then $(\mathcal{C}, \varphi)$ would be a Λ-algebra, except that $\mathcal{C}$ is in general not a set. However, if (X, p) is a Λ-algebra then a mapping $\pi : X \to \mathcal{C}$ will still be called a *homomorphism* from (X, p) to $(\mathcal{C}, \varphi)$ if

$$\pi \circ p_k = \varphi_k \circ \pi^{\vartheta(k)}$$

for all $k \in K$.

Proposition 2.4.3 *Let (X, p) be an initial Λ-algebra; then there exists a unique homomorphism $\pi : (X, p) \to (\mathcal{C}, \varphi)$.*

Proof The construction of a homomorphism from (X, p) to a Λ-algebra given in the proof of Proposition 2.4.1 takes place essentially in (X, p). This construction can thus also be used to give a homomorphism from (X, p) to $(\mathcal{C}, \varphi)$. Moreover, the uniqueness follows exactly as in the proof of Proposition 2.4.1 from the fact that (X, p) is minimal. $\square$

2.5 Free algebras

The concept of being initial will now be generalised, leading to what are known as free algebras. The existence of free algebras will be established by looking at a new class of initial algebras, which also plays an important role in the following section.

Let U be a B-family of sets, which is considered to be fixed in what follows. A Λ-algebra (X, p) is said to be *U-based* if $U \subset X$. If (X, p) and (X', p') are U-based Λ-algebras then a homomorphism $\pi : (X, p) \to (X', p')$ is said to *fix U* if $\pi_b(x) = x$ for all $x \in U_b$, $b \in B$.

Proposition 2.5.1 *(1) If (X, p) is a U-based Λ-algebra then the B-family of identity mappings* $\mathrm{id} : X \to X$ *defines a homomorphism from (X, p) to itself fixing U.*

(2) If (X,p), (Y,q) and (Z,r) are U-based Λ-algebras and $\pi : (X,p) \to (Y,q)$ and $\varrho : (Y,q) \to (Z,r)$ are homomorphisms fixing U then the composition $\varrho \circ \pi$ is a homomorphism from (X,p) to (Z,r) fixing U.

Proof This follows immediately from Proposition 2.2.1. □

Proposition 2.5.1 implies there is a category whose objects are U-based Λ-algebras and whose morphisms are homomorphisms fixing U. A Λ-algebra (X,p) is said to be *U-initial* if it is an initial object in this category, i.e., if it is U-based and if for each U-based Λ-algebra (X',p') there exists a unique homomorphism from (X,p) to (X',p') fixing U.

It will turn out that there exist U-initial Λ-algebras and that these are exactly the U-free algebras: A Λ-algebra (X,p) is said to be *U-free* if it is U-based and for each Λ-algebra (Y,q) and each B-family of mappings $v : U \to Y$ there exists a unique homomorphism $\pi^v : (X,p) \to (Y,q)$ such that $\pi^v_b(\eta) = v(\eta)$ for all $\eta \in U_b$, $b \in B$.

Proposition 2.5.2 *There exists a U-free Λ-algebra. Moreover, a Λ-algebra is U-free if and only if it is U-initial.*

Proof The proof starts by showing that there exists a U-initial Λ-algebra, and such an algebra will be obtained from an initial Λ_U-algebra, where Λ_U is the following extension of the signature Λ: For each $\eta \in U_b$, $b \in B$ let $\diamond^b_\eta$ be some element not in K and such that $\diamond^b_{\eta_1} \neq \diamond^b_{\eta_2}$ whenever $\eta_1 \neq \eta_2$ and, moreover, such that $\diamond^{b_1}_{\eta_1} \neq \diamond^{b_2}_{\eta_2}$ whenever $b_1 \neq b_2$. Put

$$K_U = K \cup \{\diamond^b_\eta : \eta \in U_b,\ b \in B\}$$

and define a signature $\Lambda_U = (B, K_U, \Theta_U)$ with $\Theta_U : K_U \to \mathcal{T}(B) \times B$ given by

$$\Theta_U(k) = \begin{cases} \Theta(k) & \text{if } k \in K, \\ (\varepsilon, b) & \text{if } k = \diamond^b_\eta \text{ for some } \eta \in U_b\ . \end{cases}$$

A U-based Λ-algebra (X,p) can be extended to a Λ_U-algebra (X,p'), called the *Λ_U-algebra associated with* (X,p), by putting $p'_k = p_k$ for each $k \in K$ and defining $p'_{\diamond^b_\eta} : \mathbb{I} \to X_b$ to be the mapping with $p'_{\diamond^b_\eta}(\varepsilon) = \eta$ for each $\eta \in U_b$, $b \in B$.

Lemma 2.5.1 *Let (X,p) and (Y,q) be U-based Λ-algebras, let (X,p') and (Y,q') be the associated Λ_U-algebras and let $\pi : X \to Y$ be a family of mappings. Then π is a homomorphism from (X,p) to (Y,q) fixing U if and only if it is a homomorphism from (X,p') to (Y,q').*

Proof This is clear. □

Lemma 2.5.2 *There exists a U-based Λ-algebra whose associated Λ_U-algebra is an initial Λ_U-algebra.*

Proof By Proposition 2.4.1 there exists an initial Λ_U-algebra, and it is easy to see that there then exists an initial Λ_U-algebra (Y, q') with $Y \cap U = \varnothing$. For each $b \in B$ put $V_b = \{q'_{\diamond^b_\eta}(\varepsilon) : \eta \in U_b\}$; then $V_b \subset Y_b$ and, since (Y, q') is regular, there is a unique bijection $\varrho_b : V_b \to U_b$ such that $\varrho_b(q'_{\diamond^b_\eta}(\varepsilon)) = \eta$ for each $\eta \in U_b$. Now put $X_b = U_b \cup (Y_b \setminus V_b)$ and extend ϱ_b to a bijection $\varrho_b : Y_b \to X_b$ by letting $\varrho_b(y) = y$ for all $y \in Y_b \setminus V_b$. Also for each $k \in K_U$ let $p'_k : X^{k^\triangleright} \to X_{k_\triangleleft}$ be given by $p'_k = \varrho_{k_\triangleleft} \circ q'_k \circ (\varrho^{-1})^{k^\triangleright}$. This results in a Λ_U-algebra (X, p'), and

$$\varrho_{k_\triangleleft} \circ q'_k = \varrho_{k_\triangleleft} \circ q'_k \circ (\varrho^{-1})^{k^\triangleright} \circ \varrho^{k^\triangleright} = p'_k \circ \varrho^{k^\triangleright}$$

for each $k \in K_U$, which means that ϱ is a homomorphism, and therefore an isomorphism, from (Y, q') to (X, p'). Hence (X, p') is itself an initial Λ_U-algebra. But by definition $(X, p) = (X, p'_{|K})$ is a U-based Λ-algebra and clearly (X, p') is the Λ_U-algebra associated with (X, p), since $p'_{\diamond^b_\eta}(\varepsilon) = \varrho_b(q'_{\diamond^b_\eta}(\varepsilon)) = \eta$ for each $\eta \in U_b$ and each $b \in B$. □

Lemma 2.5.3 *A U-based Λ-algebra is U-initial if and only if the associated Λ_U-algebra is initial.*

Proof Let (X, p) be a U-based Λ-algebra and suppose first that the associated Λ_U-algebra (X, p') is initial. Let (Y, q) be an arbitrary U-based Λ-algebra and let (Y, q') be the associated Λ_U-algebra. There is thus a unique homomorphism $\pi : (X, p') \to (Y, q')$, which by Lemma 2.5.1 is the unique homomorphism from (X, p) to (Y, q) fixing U. Thus (X, p) is a U-initial Λ-algebra.

Suppose conversely (X, p) is a U-initial Λ-algebra. By Lemma 2.5.2 there exists a U-based Λ-algebra (Y, q) whose associated Λ_U-algebra (Y, q') is initial. Let π be the unique homomorphism from (X, p) to (Y, q) fixing U and let π' be the unique Λ_U-homomorphism from (Y, q') to (X, p'). Then by Proposition 2.2.1 and Lemma 2.5.1 it follows that $\pi' \circ \pi$ is a homomorphism fixing U from (X, p) to itself and so $\pi' \circ \pi_b$ must be the family of identity mappings on X. In the same way $\pi \circ \pi'$ is a homomorphism from (Y, q') to itself and so $\pi \circ \pi'$ is the family of identity mappings on Y. This implies that π' is an isomorphism and hence (X, p'), being isomorphic to the initial Λ_U-algebra (Y, q'), is itself initial. □

In particular Lemma 2.5.3, together with Lemma 2.5.2, implies that a U-initial Λ-algebra exists. It will next be shown that U-initial Λ-algebra is U-free.

Lemma 2.5.4 *A U-initial Λ-algebra (X, p) is U-free.*

Proof By Lemma 2.5.3 the associated Λ_U-algebra (X, p') is initial. Let (Y, q) be any Λ-algebra and let $\varrho : U \to Y$ be a B-family of mappings. For each $k \in K$ let $q'_k = q_k$ and for $\eta \in U_b$ let $q'_{\diamond^b_\eta} : \mathbb{I} \to Y_b$ be given by $q'_{\diamond^b_\eta}(\varepsilon) = \varrho_b(\eta)$; thus (Y, q') is a Λ_U-algebra and so there exists a unique homomorphism π from (X, p') to (Y, q'). But then π is also a homomorphism from (X, p) to (Y, q) with

$$\pi_b(\eta) = \pi_b(p_{\diamond^b_\eta}(\varepsilon)) = q'_{\diamond^b_\eta}(\varepsilon) = \varrho_b(\eta)$$

for each $\eta \in U_b$, $b \in B$. Moreover, π is the unique such homomorphism since, conversely, any homomorphism $\pi' : (X, p) \to (Y, q)$ with $\pi'_b(\eta) = \varrho_b(\eta)$ for all $\eta \in U_b$, $b \in B$ is also a homomorphism from (X, p') to (Y, q'). Hence (X, p) is U-free. $\square$

This completes the proof of Proposition 2.5.2, since clearly a U-free Λ-algebra is U-initial. $\square$

There is a characterisation of U-initial Λ-algebras, and thus of U-free Λ-algebras, which corresponds to the second part of Proposition 2.4.1. Again let A be the parameter set of Λ.

A U-based Λ-algebra (X, p) is said to be *U-regular* if $\Im(p_k) \cap U_b = \varnothing$ for each $k \in K_b$ and if for each $b \in B \setminus A$ and each $x \in X_b \setminus U_b$ there exists a unique $k \in K_b$ and a unique element $v \in X^{k^\triangleright}$ such that $p_k(v) = x$. Thus (X, p) is U-regular if and only if the mapping p_k is injective for each $k \in K$ and for each $b \in B \setminus A$ the sets $\Im(p_k)$, $k \in K_b$, form a partition of $X_b \setminus U_b$. Similarly, a U-based Λ-algebra (X, p) is said to be *U-unambiguous* if the mapping p_k is injective for each $k \in K$ and for each $b \in B \setminus A$ the sets $\Im(p_k)$, $k \in K_b$, are disjoint subsets of $X_b \setminus U_b$.

Proposition 2.5.3 *The following are equivalent for a Λ-algebra (X, p):*

(1) (X, p) *is U-free.*
(2) (X, p) *is U-initial.*
(3) (X, p) *is U-minimal and U-regular.*
(4) (X, p) *is U-minimal and U-unambiguous.*

Proof By Proposition 2.4.1 and Lemma 2.5.3 it follows that a U-based Λ-algebra (X, p) is U-initial if and only if the associated Λ_U-algebra (X, p') is minimal and regular. But clearly (X, p') is regular if and only if $(X.p)$ is U-regular and it is minimal if and only if (X, p) is U-minimal (since a family is invariant in (X, p') if and only if it contains U and is invariant in (X, p)). Thus a Λ-algebra is U-initial if and only if it U-minimal and U-regular. This, together with Proposition 2.5.2,

shows the equivalence of (1), (2) and (3). The equivalence of (3) and (4) then follows from Proposition 2.3.1. □

If (X, p) is U-minimal then by Proposition 2.3.1 $X_{|A} = U_{|A}$ and therefore by Proposition 2.5.3 this also holds for U-free and U-initial Λ-algebras.

Lemma 2.5.5 *Let (X, p) be a U-free Λ-algebra which is an extension of an initial Λ-algebra (Z, r). Then $U \cap Z = \varnothing$.*

Proof By Proposition 2.5.3 $\Im(p_k) \subset X_b \setminus U_b$ for all $k \in K_b$, $b \in B$. But clearly $\Im(r_k) \subset \Im(p_k)$ for each $k \in K$, and so by Proposition 2.3.1 $Z_b \subset X_b \setminus U_b$ for all $b \in B$, i.e., $U \cap Z = \varnothing$. □

The following is a variation on Proposition 2.5.2:

Proposition 2.5.4 *Let (Z, r) be an initial Λ-algebra with $U \cap Z = \varnothing$. Then there exists a U-free Λ-algebra extending (Z, r). Moreover, if (X, p) and (X', p') are any two such U-free Λ-algebras and π is the unique isomorphism from (X, p) to (X', p') such that $\pi_b(\eta) = \eta$ for each $\eta \in I_b$, $b \in B$ then $\pi_b(z) = z$ for all $z \in Z_b$, $b \in B$.*

Proof By Proposition 2.5.2 there exists a U-free Λ-algebra $(\bar{X}, \bar{p})$; consider the minimal subalgebra $(\bar{Z}, \bar{r})$ of $(\bar{X}, \bar{p})$. By Proposition 2.5.3 $(\bar{Z}, \bar{r})$ is unambiguous and therefore by Proposition 2.4.1 and Lemma 2.4.1 $(\bar{Z}, \bar{r})$ is an initial Λ-algebra. Moreover, by Lemma 2.5.5 $(\bar{Z}, \bar{r})$ is disjoint from U and by construction $(\bar{X}, \bar{p})$ is an extension of $(\bar{Z}, \bar{r})$. The unique isomorphism from (Z, r) to $(\bar{Z}, \bar{r})$ can now be used to construct a U-free Λ-algebra (X, p) which is an extension of (Z, r) (and note that here the assumption is needed that $U \cap Z = \varnothing$). Finally, consider any Λ-algebra (X', p') which is an extension of (Z, r) and let π be any homomorphism from (X, p) to (X', p'). Then $\pi_b(z) = z$ for all $z \in Z_b$, $b \in B$, since the family Z' defined by $Z'_b = \{z \in Z_b : \pi_b(z) = z\}$ for each $b \in B$ is invariant in (Z, r). □

Let us close the discussion of free algebras with a result which corresponds to Proposition 2.4.3. Let $\Lambda = (K, \vartheta)$ be a single-sorted signature, let $\mathcal{C}$ be a class and let φ be a K-family of mappings with $\varphi_k : \mathcal{C}^{\vartheta(k)} \to \mathcal{C}$ for each $k \in K$. Recall that if (X, p) is a Λ-algebra then a mapping $\pi : X \to \mathcal{C}$ is still called a homomorphism from (X, p) to $(\mathcal{C}, \varphi)$ if $\pi(p_k(v)) = \varphi_k(\pi^{\vartheta(k)}(v))$ for all $v \in X^{\vartheta(k)}$, $k \in K$.

Proposition 2.5.5 *Let I be a set and $\lambda : I \to \mathcal{C}$ be a mapping. If (X, p) is an I-free Λ-algebra then there exists a unique homomorphism $\pi : (X, p) \to (\mathcal{C}, \varphi)$ such that $\pi(\eta) = \lambda(\eta)$ for each $\eta \in I$.*

Proof By Lemma 2.5.3 the Λ_I-algebra (X, p') associated with (X, p) is initial. For $k \in K$ put $\varphi'_k = \varphi_k$ and for each $\eta \in I$ define $\varphi'_{\diamond^\varepsilon_\eta} : \mathbb{I} \to \mathcal{C}$ by $\varphi'_{\diamond^\varepsilon_\eta}(\varepsilon) = \lambda(\eta)$. Hence by Proposition 2.4.3 there exists a homomorphism $\pi : (X, p') \to (\mathcal{C}, \varphi')$. But then π is also a homomorphism from (X, p) to $(\mathcal{C}, \varphi)$ with

$$\pi(\eta) = \pi(p'_{\diamond^\varepsilon_\eta}(\varepsilon)) = \varphi'_{\diamond^\varepsilon_\eta}(\varepsilon) = \lambda(\eta)$$

for each $\eta \in I$. The uniqueness follows from the fact given in Proposition 2.5.3 (3) that X is the only invariant set in (X, p) containing I. $\square$

2.6 Bound algebras

The title of this section should not be seen as having anything to do with the title of the previous section. It should be considered rather as an abbreviation of 'algebras bound to an A-family of sets'.

As before A denotes the parameter set $B \setminus \Im(\Theta_\triangleleft) = \{b \in B : K_b = \varnothing\}$ of Λ. For open signatures it is usually the case that an A-family of sets V is given and the interest is then only in Λ-algebras (X, p) with $X_{|A} = V$. This is the situation which will be dealt with here.

Let V be an A-family of sets, which is considered to be fixed in what follows. A Λ-algebra (X, p) is said to be *bound to* V if $X_{|A} = V$. Of course, if Λ is closed (i.e., if $A = \varnothing$) then there is only one A-family of sets and any Λ-algebra is bound to it. If (X, p) and (X', p') are Λ-algebras bound to V then a homomorphism $\pi : (X, p) \to (X', p')$ is said to *fix* V if $\pi_a(x) = x$ for each $x \in V_a$, $a \in A$. Again, if Λ is closed then this imposes no requirement on a homomorphism.

Proposition 2.6.1 *(1) If (X, p) is a Λ-algebra bound to V then the B-family of identity mappings* $\mathrm{id} : X \to X$ *defines a homomorphism from (X, p) to itself fixing V.*

(2) If (X, p), (Y, q) and (Z, r) are Λ-algebras bound to V and $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are homomorphisms fixing V then the composition $\varrho \circ \pi$ is a homomorphism from (X, p) to (Z, r) fixing V.

Proof This follows immediately from Proposition 2.2.1. $\square$

Proposition 2.6.1 implies there is a category whose objects are Λ-algebras bound to V and whose morphisms are homomorphisms fixing V. A Λ-algebra (X, p) is called *V-initial* if it is an initial object in this category, i.e., if it is bound to V and for each Λ-algebra (X', p') bound to V there exists a unique homomorphism $\pi : (X, p) \to (X', p')$ fixing V.

Proposition 2.6.2 *There exists a V-initial Λ-algebra.*

Proof For the rest of the section let $\bar{V}$ be the B-family of sets with $\bar{V}_{|A} = V$ and $\bar{V}_{|B\setminus A} = \varnothing$; this B-family will be called the *trivial extension of* V *to* B. Note that any Λ-algebra bound to V is then $\bar{V}$-based. To show the existence of a V-initial Λ-algebra part of the following result is needed (the remainder being required for the proof of Proposition 2.6.3 below).

Lemma 2.6.1 *An initial $\bar{V}$-based Λ-algebra is V-initial. Moreover, a Λ-algebra is V-initial if and only if it is $\bar{V}$-initial.*

Proof Proposition 2.5.3 and Proposition 2.3.2 imply that a $\bar{V}$-initial Λ-algebra is bound to V, and thus is a V-initial Λ-algebra (since each Λ-algebra bound to V is $\bar{V}$-based, and fixing $\bar{V}$ is of course the same as fixing V). This also shows that a Λ-algebra bound to V and which is a $\bar{V}$-initial Λ-algebra is V-initial. Conversely, let (X, p) be V-initial, and let (X', p') be a $\bar{V}$-initial Λ-algebra (whose existence is guaranteed by Proposition 2.5.3). By the first part of the proof (X', p') is then V-initial. Therefore (X, p) and (X', p') are isomorphic as Λ-algebras bound to V, and so they are also isomorphic as $\bar{V}$-based Λ-algebras. This implies then that (X, p) is itself $\bar{V}$-initial. □

The existence of a V-initial Λ-algebra now follows from Proposition 2.5.3 and the first statement in Lemma 2.6.1. □

For the situation being considered here there is a result which corresponds to Proposition 2.4.2. A Λ-algebra (X, p) bound to V is said to be *V-minimal* if X is the only invariant family Y in (X, p) such that $Y_{|A} = V$.

Lemma 2.6.2 *A V-minimal Λ-algebra is $\bar{V}$-minimal. Conversely, a $\bar{V}$-minimal Λ-algebra bound to V is V-minimal.*

Proof Let (X, p) be a Λ-algebra bound to V. Then an invariant family Y satisfies $Y_{|A} = V$ if and only if it contains $\bar{V}$. This implies that (X, p) is V-minimal if and only if it is $\bar{V}$-minimal. The result thus follows because a V-minimal Λ-algebra is bound to V by definition. □

Proposition 2.6.3 *The following are equivalent for a Λ-algebra (X, p):*

(1) (X, p) *is V-initial.*
(2) (X, p) *is V-minimal and regular.*
(3) (X, p) *is V-minimal and unambiguous.*

Proof Let (X, p) be a Λ-algebra bound to V. Then by Lemma 2.6.2 (X, p) is V-minimal if and only if it is $\bar{V}$-minimal. Furthermore, (since $\bar{V}_b = \varnothing$ for each $b \in B \setminus A$) (X, p) is regular if and only if it is $\bar{V}$-regular and unambiguous if and only if it is $\bar{V}$-unambiguous. The result thus follows from Proposition 2.5.3 and the second statement in Lemma 2.6.1. □

Proposition 2.6.4 *If (X, p) is a V-minimal Λ-algebra then $X_b = \bigcup_{k \in K_b} \Im(p_k)$ for each $b \in B \setminus A$. Moreover, a graded Λ-algebra (X, p) is V-minimal if and only if $X_b = \bigcup_{k \in K_b} \Im(p_k)$ for each $b \in B \setminus A$.*

Proof This follows from Propositions 2.3.2 and 2.3.3 and Lemma 2.6.2. □

There is an obvious grading for the Λ-algebra (X, p) defined in Example 2.2.3, and it is thus easy to check that this algebra is V-minimal. It then follows that (X, p) is V-initial, since it is also clearly unambiguous.

Note that if (X, p) is a V-initial Λ-algebra and $b \in B \setminus A$ is a primitive type then by Proposition 2.6.3 the mapping $k \mapsto p_k(\varepsilon)$ maps K_b bijectively onto X_b. (For the Λ-algebra (X, p) in Example 2.2.3 with primitive types `atom`, `bool` and `int` and this gives the obvious bijections between $K_{\mathtt{atom}} = \{\mathtt{Atom}\}$ and $X_{\mathtt{atom}} = \mathbb{I}$, between $K_{\mathtt{bool}} = \{\mathtt{True}, \mathtt{False}\}$ and $X_{\mathtt{bool}} = \mathbb{B} = \{\mathrm{T}, \mathrm{F}\}$ and between $K_{\mathtt{int}} = \underline{Int}$ and $X_{\mathtt{int}} = \mathbb{Z}$.)

It turns out that a V-initial Λ-algebra has a seemingly stronger property than that of just being V-initial: A Λ-algebra (X, p) will be called *intrinsically free* if for each Λ-algebra (Y, q) and each A-family of mappings $\varrho : X_{|A} \to Y_{|A}$ there exists a unique homomorphism $\pi : (X, p) \to (Y, q)$ such that $\pi_{|A} = \varrho$.

Proposition 2.6.5 *A Λ-algebra (X, p) bound to V is V-initial if and only if it is intrinsically free.*

Proof An intrinsically free Λ-algebra bound to V is clearly V-initial and the converse follows immediately from Lemma 2.6.1 and Lemma 2.5.4. □

Proposition 2.6.6 *Let (X, p) be a Λ-algebra and let U be an A-family of sets with $U \subset X_{|A}$. Then there exists a unique U-minimal subalgebra $(\hat{X}, \hat{p})$ of (X, p).*

Proof Let $(\hat{X}, \hat{p})$ be the minimal subalgebra of (X, p) containing $\bar{U}$ (the trivial extension of U to B). Then by Lemma 2.3.4 $\hat{X}_{|A} = U$, since $K_a = \varnothing$ for each $a \in A$, i.e., $(\hat{X}, \hat{p})$ is bound to U. Moreover, $(\hat{X}, \hat{p})$ is U-minimal, since if X' is an invariant family in $(\hat{X}, \hat{p})$ with $X'_{|A} = \hat{X}_{|A}$ then X' is an invariant family in (X, p) containing $\bar{U}$ and hence $X' = \hat{X}$. Finally, if (Y, q) is any U-minimal subalgebra of (X, p) bound to U then the B-family $X' = \hat{X} \cap Y$ is invariant in (X, p) and contains $\bar{U}$. Thus $Y = X' = \hat{X}$, which gives the uniqueness of $(\hat{X}, \hat{p})$. □

Lemma 2.6.3 *A Λ-algebra is initial if and only if it is $\varnothing$-initial.*

Proof First note that a homomorphism between Λ-algebras bound to $\varnothing$ trivially fixes $\varnothing$. Moreover, as already mentioned above, an initial Λ-algebra is bound to $\varnothing$, and therefore it is $\varnothing$-initial. Conversely, let (X, p) be $\varnothing$-initial and consider any Λ-algebra (Y, q). Now the family Y' with $Y'_{|B\setminus A} = Y_{|B\setminus A}$ and $Y'_{|A} = \varnothing$ is invariant in (Y, q) and the associated subalgebra (Y', q') is bound to $\varnothing$, there thus exists a unique homomorphism $\pi : (X, p) \to (Y', q')$. But π is then the unique homomorphism from (X, p) to (Y, q) (it being unique because any homomorphism from (X, p) to (Y, q) is also a homomorphism from (X, p) to (Y', q')). Hence (X, p) is initial. $\square$

2.7 Term algebras

This section introduces what are called term algebras. These provide the simplest explicit examples of initial algebras, and they can be used as the basic components of a 'real' programming language. The main result (Proposition 2.7.1) is really just a version of the classical result of Łukasiewicz [10] concerning prefix or (left Polish) notation. Until further notice assume that Λ is an enumerated signature; the general case is dealt with at the end of the section.

Let Z be a set; then the concatenation of two lists ℓ, $\ell' \in Z^*$ will be denoted by $\ell\,\ell'$. Thus $\ell\,\varepsilon = \varepsilon\,\ell = \ell$ and if $\ell = z_1 \cdots z_m$, $\ell' = z'_1 \cdots z'_n$ with m, $n \geq 1$ then

$$\ell\,\ell' = z_1 \cdots z_m\, z'_1 \cdots z'_n \ .$$

Concatenation is clearly associative, and hence if $p \geq 1$ and $\ell_1, \ldots, \ell_p \in Z^*$ then *the* concatenation of $\ell_1, \ldots, \ell_p$ can be denoted simply by $\ell_1 \cdots \ell_p$. This notation is clearly compatible with the notation being employed for the elements of Z^* (in the sense that $\ell = z_1 \cdots z_n \in Z^*$ can be considered as the concatenation of the n one element lists $z_1, \ldots, z_n$). If $z \in Z$ and $\ell \in Z^*$ then in particular there is the list $z\,\ell$ obtained by adding z to the beginning of the list ℓ (a list which is also being denoted by $z \triangleleft \ell$).

Let Ω be a set and $\Gamma : K \to \Omega$ be a mapping. A Λ-algebra (Y, q) can then be obtained as follows: For each $b \in B$ put $Y_b = \Omega^*$; if $k \in K$ is of type $b_1 \cdots b_n \to b$ (with $n \geq 0$) then let $q_k : Y_{b_1} \times \cdots \times Y_{b_n} \to Y_b$ be the mapping defined by

$$q_k(y_1, \ldots, y_n) = \Gamma(k)\, y_1 \cdots y_n$$

for each $(y_1, \ldots, y_n) \in Y_{b_1} \times \cdots \times Y_{b_n}$, i.e., $q_k(y_1, \ldots, y_n)$ is the list obtained by concatenating the element $\Gamma(k)$ and the lists $y_1, \ldots, y_n$. In particular, if $k \in K$ is of type $\varepsilon \to b$ then $q_k(\varepsilon)$ is just the list consisting of the single component

$\Gamma(k)$. (This is really just an instance of the construction described at the end of Section 2.2.)

Now let $(E, \Box)$ be the minimal subalgebra of (Y, q), i.e., E is the minimal invariant family and $\Box_k$ is the corresponding restriction of q_k for each $k \in K$. This minimal Λ-algebra $(E, \Box)$ is called the *term* Λ*-algebra specified by* Γ and Γ is then referred to as a *term algebra specifier*. By Proposition 2.3.2 it is clear that each element of E_b contains at least one component, i.e., $\varepsilon \notin E_b$ for each $b \in B$.

The simplest example of this construction is with $\Omega = K$ and with $\Gamma : K \to K$ the identity mapping: The term Λ-algebra specified by this mapping is called the *standard term* Λ*-algebra*.

It turns out that the standard term Λ-algebra is initial. This is a special case of Proposition 2.7.1 below.

Let $\Gamma : K \to \Omega$ be a term algebra specifier and $(E, \Box)$ be the term Λ-algebra specified by Γ. For each $k \in K$ let $\chi_k : E^{k^\triangleright} \times \Omega^* \to \Omega^*$ be defined by

$$\chi_k(v, \alpha) = \Box_k(v)\, \alpha\ .$$

Lemma 2.7.1 *Suppose* $\Im(\chi_{k_1})$ *and* $\Im(\chi_{k_2})$ *are disjoint subsets of* Ω^* *whenever* $b \in B$ *and* $k_1,\ k_2 \in K_b$ *with* $k_1 \neq k_2$*. Then* $(E, \Box)$ *is initial.*

Proof By Proposition 2.4.2 it is enough to show $(E, \Box)$ is unambiguous. But

$$\Im(\Box_k) = \chi_k(E^{k^\triangleright} \times \{\varepsilon\}) \subset \Im(\chi_k)$$

for each $k \in K$; thus if $k_1,\ k_2 \in K_b$ with $k_1 \neq k_2$ then $\Im(\Box_{k_1})$ and $\Im(\Box_{k_2})$ are clearly disjoint. It therefore remains to show that $\Box_k$ is injective for each $k \in K$, and for this the following lemma is required:

Lemma 2.7.2 *Suppose that the assumption in the statement of Lemma 2.7.1 is satisfied and let* $\chi_b : E_b \times \Omega^* \to \Omega^*$ *be defined by* $\chi_b(e, \alpha) = e\,\alpha$*. Then for each* $b \in B$ *the mapping* χ_b *is injective.*

Proof For each $b \in B$ define a subset G_b of Ω^* by

$$G_b = \{e \in E_b : e\,\alpha \in E_b \text{ for some } \alpha \in \Omega^* \text{ with } \alpha \neq \varepsilon\}$$

and put $G = \bigcup_{b \in B} G_b$. Then, since χ_b is injective if and only if $G_b = \varnothing$, it follows that $G \neq \varnothing$ if and only if χ_b is not injective for some $b \in B$.

Suppose that $G \neq \varnothing$ and let $m = \min\{|\alpha| : \alpha \in G\}$, where $|\alpha|$ denotes the number of components in the list $\alpha \in \Omega^*$. There thus exists $b \in B$, $e \in E_b$ and $\alpha \in \Omega^* \setminus \{\varepsilon\}$ such that $|e| = m$ and $e' = e\,\alpha \in E_b$. Now by Proposition 2.3.2

there exist k, $k' \in K_b$ and v, v' such that $e = \Box_k(v)$ and $e' = \Box_{k'}(v')$, and this implies that $\chi_k(v,\alpha) = e\,\alpha = e' = e'\,\varepsilon = \chi_{k'}(v',\varepsilon)$, which by assumption is only possible if $k' = k$. Let k be of type $b_1 \cdots b_n \to b$ and put $\Gamma(k) = \alpha'$. Thus if $v = (e_1, \ldots, e_n)$ and $v' = (e'_1, \ldots, e'_n)$ then $e = \alpha'\, e_1 \cdots e_n$ and $e\,\alpha = \alpha'\, e'_1 \cdots e'_n$. There must therefore exist $1 \le j \le n$ with $e_j \ne e'_j$ (and so in particular $n > 0$). Let i be the least index with $e_i \ne e'_i$ and let $\hat{e}$ be the shorter of the lists e_i and e'_i. But then $\hat{e} \in G_{b_i}$ and $|\hat{e}| < |e|$, which by the minimality of $|e|$ is not possible.

This shows that $G = \varnothing$ and hence that χ_b is injective for each $b \in B$. $\Box$

The proof of Lemma 2.7.1 can now be completed. Thus let $k \in K$ be of type $b_1 \cdots b_n \to b$ and let $v, v' \in E^{k^\triangleright}$ with $\Box_k(v) = \Box_k(v')$. This means that if $v = (e_1, \ldots, e_n)$, $v' = (e'_1, \ldots, e'_n)$ then $\Gamma(k)\, e_1 \cdots e_n = \Gamma(k)\, e'_1 \cdots e'_n$. Consider $1 \le j \le n$ and assume $e_i = e'_i$ for each $i < j$. Then

$$\chi_{b_j}(e_j, e_{j+1} \cdots e_n) = e_j\, e_{j+1} \cdots e_n = e'_j\, e'_{j+1} \cdots e'_n = \chi_{b_j}(e'_j, e'_{j+1} \cdots e'_n)$$

and so by Lemma 2.7.1 $e_j = e'_j$. Therefore by n applications of Lemma 2.7.2 it follows that $e_j = e'_j$ for each $j = 1, \ldots, n$; i.e., $v = v'$. This shows that $\Box_k$ is injective. $\Box$

The term algebra specifier $\Gamma : K \to \Omega$ will be called *locally injective* if for each $b \in B$ the restriction of Γ to K_b is injective. (Of course, since $K_{|A} = \varnothing$, this is equivalent to requiring that the restriction of Γ to K_b be injective for each $b \in B \setminus A$.)

Proposition 2.7.1 *The term Λ-algebra $(E, \Box)$ specified by a locally injective specifier Γ is initial.*

Proof This follows immediately from the fact that $\Gamma(k)$ is the first component of each element of $\Im(\chi_k)$, and hence the hypothesis of Lemma 2.7.1 is satisfied. $\Box$

Proposition 2.7.2 *The standard term Λ-algebra is initial.*

Proof This follows from Proposition 2.7.1, since the identity mapping used to specify the standard term Λ-algebra is trivially locally injective. $\Box$

If $(E, \Box)$ is initial then the family E can be thought of, somewhat informally, as being defined by the following rules:

(1) If $k \in K$ is of type $\varepsilon \to b$ then the list consisting of the single component $\Gamma(k)$ is an element of E_b.

(2) If $k \in K$ is of type $b_1 \cdots b_n \to b$ with $n \ge 1$ and $e_j \in E_{b_j}$ for each j then $\Gamma(k)\, e_1 \cdots e_n$ is an element of E_b.

(3) Each element of E_b can be obtained in a unique way using finitely many applications of rules (1) and (2).

These rules are just meant to be a rather imprecise statement of Proposition 2.4.1. In the special case when $(E, \square)$ is the standard term Λ-algebra then $E_b \subset K^*$ for each $b \in B$ and the family E is defined by the following rules:

(1) If $k \in K$ is of type $\varepsilon \to b$ then the list consisting of the single component k is an element of E_b.

(2) If $k \in K$ is of type $b_1 \cdots b_n \to b$ with $n \geq 1$ and $e_j \in E_{b_j}$ for each j then $k\, e_1 \cdots e_n$ is an element of E_b.

(3) Each element of E_b can be obtained in a unique way using finitely many applications of rules (1) and (2).

Example 2.7.1 Consider the standard term Λ-algebra $(E, \square)$ arising from the enumerated signature Λ in Example 2.2.1. Then:

$E_{\texttt{bool}}$ consists of the two elements `True` and `False` of K^*.
$E_{\texttt{nat}}$ consists of exactly the following elements of K^*:

$$\texttt{Zero},\ \texttt{Succ Zero},\ \texttt{Succ Succ Zero},\ \texttt{Succ Succ Succ Zero},\ \ldots\ .$$

$E_{\texttt{int}}$ consists of all elements of K^* having the form $\underline{n}$ with $n \in \mathbb{Z}$.
$E_{\texttt{pair}}$ consists of all elements of K^* of the form $\texttt{Pair}\,\underline{m}\,\underline{n}$ with $m,\ n \in \mathbb{Z}$. (Each element of $E_{\texttt{pair}}$ thus has exactly three components.)
$E_{\texttt{list}}$ consists of the element `Nil` plus all elements of K^* having the form $\texttt{Cons}\,\underline{n}\,e$ with $n \in \mathbb{Z}$ and $e \in E_{\texttt{list}}$. For instance,

$$\texttt{Cons 42 Cons } -128 \texttt{ Cons 0 Cons } -21 \texttt{ Nil}$$

is an element of $E_{\texttt{list}}$.

It will now no longer be assumed that Λ is enumerated. In order to apply the above results to Λ the problem of replacing the general signature $\Lambda = (B, K, \Theta)$ with an ‘equivalent’ enumerated signature $\Lambda' = (B, K, \Theta')$ must be considered. In order to carry this out fix for each $k \in K$ an enumeration of the elements in the set $\langle k^{\triangleright} \rangle = \mathrm{dom}(k^{\triangleright})$: More precisely, choose a bijective mapping i_k from $[n_k]$ to $\langle k^{\triangleright} \rangle$, where n_k is the cardinality of $\langle k^{\triangleright} \rangle$. There is then a mapping $k \mapsto k^{\circ}$ from K to B^* given by $k^{\circ} = k^{\triangleright} \circ i_k$ for each $k \in K$, i.e., with

$$k^{\circ} = k^{\triangleright}(i_k(1)) \cdots k^{\triangleright}(i_k(n_k))\ ,$$

and the mapping $\Theta' : K \to B^* \times B$ with $\Theta' = (k^\circ, k_\triangleleft)$. This defines an enumerated signature $\Lambda' = (B, K, \Theta')$, which will be called the *signature obtained from* Λ *and the family of enumerations* i.

Lemma 2.7.3 *Let* X *be a family of sets and let* $k \in K$ *with* $k^\circ = b_1 \cdots b_{n_k}$. *Then the mapping* $i_k^* : X^{k^\triangleright} \to X_{b_1} \times \cdots \times X_{b_{n_k}}$ *given by*

$$i_k^*(v) = (v(i_k(1)), \ldots, v(i_k(n_k)))$$

for each $v \in X^{k^\triangleright}$ *is a bijection.*

Proof This is clear. □

Proposition 2.7.3 *Let* (X, p') *be a* Λ'*-algebra and put* $p = p' \circ i^*$. *Then* (X, p) *is a* Λ*-algebra. Conversely, if* (X, p) *is any* Λ*-algebra then there exists a unique* Λ'*-algebra* (X, p') *such that* $p = p' \circ i^*$.

Proof This follows immediately from Lemma 2.7.3. □

If (X, p') is a Λ'-algebra and $p = p' \circ i^*$ then (X, p) is called the Λ-*algebra associated with* (X, p') *and* i.

Proposition 2.7.3 says there is a one-to-one correspondence between Λ-algebras and Λ'-algebras. Moreover, Proposition 2.7.4 below implies that a Λ'-algebra has a property (such as being minimal or initial) if and only if the corresponding Λ-algebra also has this property. In this sense the signatures Λ' and Λ can be regarded as being equivalent.

Proposition 2.7.4 *Let* (X, p') *be a* Λ'*-algebra and let* (X, p) *be the* Λ*-algebra associated with* (X, p') *and* i. *Then:*

(1) (X, p) *is minimal if and only if* (X, p') *is a minimal* Λ'*-algebra.*

(2) (X, p) *is initial if and only if* (X, p') *is an initial* Λ'*-algebra.*

Proof (1) This follows from the easily verified fact that a family $\grave{X}$ is invariant in (X, p') if and only if it is invariant in (X, p).

(2) This follows from (1) and Proposition 2.4.2. □

Now let $\Gamma : K \to \Omega$ be a mapping, which will still be referred to as a *term algebra specifier*. Then Γ is also a term algebra specifier for the signature Λ', so let $(E, \Box')$ be the term Λ'-algebra specified by Γ. Put $\Box = \Box' \circ i^*$ (with i^* given by Lemma 2.7.3). This means that $E_b \subset \Omega^*$ for each $b \in B$ and for each $k \in K$ the mapping $\Box_k : E^{k^\triangleright} \to E_b$ is given by

$$\Box_k(v) = \Gamma(k)\, v(i_k(1)) \cdots v(i_k(n_k))$$

for each $v \in E^{k^\triangleright}$. Moreover, the family E can be regarded as being defined by the following rules:

(1) If $k \in K$ with $k^{\triangleright} = \varepsilon$ then the list consisting of the single component $\Gamma(k)$ is an element of $E_{k_{\triangleleft}}$.

(2) If $k \in K$ with $k^{\triangleright} \neq \varepsilon$ and $e_j \in E_{b_j}$ for $j = 1, \ldots, n_k$, where $b_j = k^{\triangleright}(i_k(j))$, then $\Gamma(k)\, e_1 \cdots e_{n_k}$ is an element of $E_{k_{\triangleleft}}$.

(3) Each element of E_b can be obtained in a unique way using finitely many applications of rules (1) and (2).

The algebra $(E, \sqsubset)$ is called the *term* Λ*-algebra specified by* Γ *and the family of enumerations* i, or simply the *standard term* Λ*-algebra defined by the family* i in the special case when $\Gamma : K \to K$ is the identity mapping.

Lemma 2.7.4 *The* Λ*-algebra* $(E, \sqsubset)$ *is initial if and only if* $(E, \sqsubset')$ *is an initial* Λ'*-algebra.*

Proof This follows from Proposition 2.7.4 (2), since by definition $(E, \sqsubset)$ is the Λ-algebra associated with $(E, \sqsubset')$ and i. $\square$

Proposition 2.7.5 *If* Γ *is locally injective then* $(E, \sqsubset)$ *is an initial* Λ*-algebra. In particular, the standard term* Λ*-algebra defined by any family of enumerations is initial.*

Proof This follows immediately from Proposition 2.7.1 and Lemma 2.7.4. $\square$

Of course, if the signature Λ is enumerated and the above constructions are applied with the family of identity enumerations i then the end-result is that nothing happens, since in this case $\Lambda' = \Lambda$.

The final topic of this section considers the construction of term algebras when a further signature $\Lambda' = (B', K', \Theta')$ is given which is an extension of Λ. Suppose in what follows that $\Gamma' : K' \to \Omega'$ is a term algebra specifier which is an *extension* of a term algebra specifier $\Gamma : K \to \Omega$ in the sense that $\Omega \subset \Omega'$ and $\Gamma(k) = \Gamma'(k)$ for each $k \in K$.

Lemma 2.7.5 *Assume that the signatures* Λ *and* Λ' *are both enumerated. Then the term* Λ'*-algebra specified by* Γ' *is an extension of the term* Λ*-algebra* $(E, \sqsubset)$ *specified by* Γ.

Proof This follows from Lemma 2.7.6, because the Λ'-algebra corresponding to the Λ-algebra (Y, q) in the definition of $(E, \sqsubset)$ is an extension of (Y, q). $\square$

Lemma 2.7.6 *Let (Y, q) be a Λ'-algebra which is an extension of a Λ-algebra (X, p), and let $\hat{X}$ be the minimal invariant family in (X, p) and $\hat{Y}$ the minimal invariant family in (Y, q). Then $\hat{X} \subset \hat{Y}_{|B}$.*

Proof Let $\grave{Y}$ be any invariant family in (Y, q) and put $Z = \grave{Y}_{|B} \cap X$. Then the family Z is invariant in (X, p), since

$$p_k(Z^{k^{\triangleright}}) \subset q_k(\grave{Y}^{k^{\triangleright}}) \cap X_{k_{\triangleleft}} \subset \grave{Y}_{k_{\triangleleft}} \cap X_{k_{\triangleleft}} = Z_{k_{\triangleleft}}$$

for each $k \in K$. Thus $\hat{X} \subset Z$ and hence $\hat{X} \subset \grave{Y}$. In particular, $\hat{X} \subset \hat{Y}_{|B}$. $\square$

Lemma 2.7.6 implies that the standard term Λ'-algebra defined by the family of enumerations i' is an extension of the standard term Λ-algebra defined by the family i.

Chapter 3

Bottomed algebras

With the preparations made in Chapter 2, the next step in building a framework for specifying data objects will now be introduced. This involves the treatment of 'undefined' and 'partially defined' objects.

In any programming language data objects are manipulated by algorithms, and in any non-trivial language it is an unavoidable fact that algorithms need not terminate. It is thus necessary to introduce an 'undefined' element for each type in order to represent this state of affairs. Moreover, depending on the language, it may also be necessary to have 'partially defined' data objects, for example a pair whose first component is defined but not the second, or a list in which only some of the components are defined. To deal with this situation bottomed algebras will be introduced. These are algebras containing for each type $b \in B$ a special bottom element $\perp_b$ to denote an 'undefined' element of the type.

3.1 Bottomed algebras and homomorphisms

Let $\Lambda = (B, K, \Theta)$ be a signature which is considered to be fixed throughout the chapter; A will always denote the parameter set of Λ, i.e., $A = \{b \in B : K_b = \varnothing\}$. A *bottomed* Λ*-algebra* is any pair (X, p) consisting of a B-family of bottomed sets X and a K-family of mappings p such that p_k is a mapping from $X^{k^{\triangleright}}$ to $X_{k_{\triangleleft}}$ for each $k \in K$. If (X, p) is a bottomed Λ-algebra and $\check{X}_b$ is the underlying set of X_b for each $b \in B$, then $(\check{X}, p)$ is an 'ordinary' Λ-algebra which will be called the *underlying* Λ*-algebra* or just the *underlying algebra*. Following the convention made in Section 2.1 X_b will also be used to denote the underlying set $\check{X}_b$, which means that the underlying and the bottomed Λ-algebra will both be denoted by (X, p).

If (X, p) is a bottomed Λ-algebra then the bottom element of X_b will (almost) always be denoted by $\perp_b$. Thus if (Y, q) is a further bottomed Λ-algebra then

the bottom element of Y_b will also be denoted by $\perp_b$, although it is not to be assumed that the bottom elements of X_b and Y_b are the same. Recall that if Z is a bottomed set with bottom element $\perp$ then $Z^\natural$ denotes the set $Z \setminus \{\perp\}$; thus if (X, p) is a bottomed Λ-algebra then $X_b^\natural$ is the set $X_b \setminus \{\perp_b\}$ for each $b \in B$.

If (Y, q) is a Λ-algebra then a bottomed Λ-algebra (X, p) will be called a *bottomed extension* of (Y, q) if the underlying algebra of (X, p) is an extension of (Y, q) and $\perp_b \notin Y_b$ for each $b \in B$. The simplest way to obtain bottomed Λ-algebras is a special case of a such an extension: Let (Y, q) be any Λ-algebra and for each $b \in B$ choose an element $\perp_b$ not in Y_b. For each $b \in B$ put $Y_b^\perp = Y_b \cup \{\perp_b\}$, and consider $Y_b^\perp$ as a bottomed set with bottom element $\perp_b$. For each $k \in K$ let $q_k^\perp : (Y^\perp)^{k^\triangleright} \to Y_{k_\triangleleft}^\perp$ be given by

$$q_k^\perp(v) = \begin{cases} q_k(v) & \text{if } v \in Y^{k^\triangleright}, \\ \perp_{k_\triangleleft} & \text{otherwise.} \end{cases}$$

Then $(Y^\perp, q^\perp)$ is a bottomed extension called the *flat bottomed extension* of (Y, q).

A bottomed Λ-algebra (X, p) should be thought of a describing all data objects, both the basic defined objects and those in various degrees of not being defined. In particular, (X, p) will be a bottomed extension of a Λ-algebra (Y, q) describing the defined data objects. However, it simplifies things to only work with (X, p) without continually having to worry about (Y, q). On the other hand, it only makes sense to work with bottomed Λ-algebras which are bottomed extensions of suitable Λ-algebras. There is a simple condition ensuring that this is the case which will be introduced in the the following section (since it fits in better with the material considered there).

If (X, p) and (Y, q) are bottomed Λ-algebras then a *bottomed homomorphism* $\pi : (X, p) \to (Y, q)$ is a homomorphism of the underlying algebras such that the mappings in the family π are all bottomed, i.e., $\pi_b(\perp_b) = \perp_b$ for each $b \in B$.

We consider a set-up including the situation typical for open signatures, in which an A-family of bottomed sets V is given and the interest is then only in bottomed Λ-algebras (X, p) with $X_{|A} = V$.

Let V be an A-family of bottomed sets, which is considered to be fixed in what follows. A bottomed Λ-algebra (X, p) is said to be *bound to* V if $X_{|A} = V$. Of course, if Λ is closed (i.e., if $A = \varnothing$) then there is only one A-family of bottomed sets and any bottomed Λ-algebra is bound to it.

Let U be an A-family of sets and let (Y, q) be a Λ-algebra fixing U. Then the flat bottomed extension $(Y^\perp, q^\perp)$ of (Y, q) is a bottomed Λ-algebra fixing $U^\perp$, where $U_a^\perp = U_a \cup \{\perp_a\}$ (considered as a bottom set with bottom element $\perp_a$) for each $a \in A$. Conversely, a simple way to obtain a bottomed Λ-algebra fixing V is to construct a Λ-algebra (Y, q) fixing $V^\natural$; the flat bottomed extension of (Y, q) is then a bottomed Λ-algebra fixing V. In Example 3.1.1 this method is employed

to obtain a bottomed Λ-algebra fixing a given A-family of bottomed sets with Λ the signature in Example 2.2.3. (The Λ-algebra (Y, q) used in the example is essentially that occurring in Example 2.2.3.)

Example 3.1.1 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 2.2.3 (so $A = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$) and let V be an A-family of bottomed sets. Then a bottomed Λ-algebra (Y, q) bound to V can be defined by letting

$Y_{\mathtt{bool}} = \mathbb{B}^{\perp} = \mathbb{B} \cup \{\perp_{\mathtt{bool}}\}$, $Y_{\mathtt{atom}} = \mathbb{I}^{\perp} = \mathbb{I} \cup \{\perp_{\mathtt{atom}}\}$,
$Y_{\mathtt{int}} = \mathbb{Z}^{\perp} = \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}$, $Y_{\mathtt{pair}} = (V_{\mathtt{x}}^{\natural} \times V_{\mathtt{y}}^{\natural}) \cup \{\perp_{\mathtt{pair}}\}$,
$Y_{\mathtt{list}} = (V_{\mathtt{z}}^{\natural})^* \cup \{\perp_{\mathtt{list}}\}$, $Y_{\mathtt{lp}} = (V_{\mathtt{x}}^{\natural} \times V_{\mathtt{y}}^{\natural}) \cup (V_{\mathtt{z}}^{\natural})^* \cup \{\perp_{\mathtt{lp}}\}$,
$Y_{\mathtt{x}} = V_{\mathtt{x}}$, $Y_{\mathtt{y}} = V_{\mathtt{y}}$, $Y_{\mathtt{z}} = V_{\mathtt{z}}$,

all these unions being considered to be disjoint and Y_b being considered as a bottomed set with bottom element $\perp_b$ for each $b \in B \setminus A$,

$q_{\mathtt{True}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{True}}(\varepsilon) = \mathrm{T}$,
$q_{\mathtt{False}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{False}}(\varepsilon) = \mathrm{F}$,
$q_{\mathtt{Atom}} : \mathbb{I} \to Y_{\mathtt{atom}}$ with $q_{\mathtt{Atom}}(\varepsilon) = \varepsilon$,
$q_{\underline{n}} : \mathbb{I} \to Y_{\mathtt{int}}$ with $q_{\underline{n}}(\varepsilon) = n$ for each $n \in \mathbb{Z}$,
$q_{\mathtt{Pair}} : Y_{\mathtt{x}} \times Y_{\mathtt{y}} \to Y_{\mathtt{pair}}$ with

$$q_{\mathtt{Pair}}(x, y) = \begin{cases} (x, y) & \text{if } x \in V_{\mathtt{x}}^{\natural} \text{ and } y \in V_{\mathtt{y}}^{\natural}, \\ \perp_{\mathtt{pair}} & \text{otherwise,} \end{cases}$$

$q_{\mathtt{Nil}} : \mathbb{I} \to Y_{\mathtt{list}}$ with $q_{\mathtt{Nil}}(\varepsilon) = \varepsilon$,
$q_{\mathtt{Cons}} : Y_{\mathtt{z}} \times Y_{\mathtt{list}} \to Y_{\mathtt{list}}$ with

$$q_{\mathtt{Cons}}(z, s) = \begin{cases} m \triangleleft s & \text{if } z \in V_{\mathtt{z}}^{\natural} \text{ and } s \in (V_{\mathtt{z}}^{\natural})^*, \\ \perp_{\mathtt{list}} & \text{otherwise,} \end{cases}$$

$$q_{\mathtt{L}} : Y_{\mathtt{list}} \to Y_{\mathtt{lp}} \text{ with } q_{\mathtt{L}}(s) = \begin{cases} s & \text{if } s \in (V_{\mathtt{z}}^{\natural})^*, \\ \perp_{\mathtt{lp}} & \text{otherwise,} \end{cases}$$

$$q_{\mathtt{P}} : Y_{\mathtt{pair}} \to Y_{\mathtt{lp}} \text{ with } q_{\mathtt{P}}(p) = \begin{cases} p & \text{if } p \in V_{\mathtt{x}}^{\natural} \times V_{\mathtt{y}}^{\natural}, \\ \perp_{\mathtt{lp}} & \text{otherwise.} \end{cases}$$

If (X, p) and (X', p') are bottomed Λ-algebras bound to V then a bottomed homomorphism $\pi : (X, p) \to (X', p')$ is said to *fix* V if $\pi_a(x) = x$ for each $x \in V_a$, $a \in A$. Again, if Λ is closed then this imposes no requirement on a bottomed homomorphism.

Proposition 3.1.1 *(1) If (X, p) is a bottomed Λ-algebra bound to V then the B-family of identity mappings* id : $X \to X$ *defines a bottomed homomorphism from (X, p) to itself fixing V.*

(2) If $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are bottomed homomorphisms fixing V then the composition $\varrho \circ \pi$ is a bottomed homomorphism from (X, p) to (Z, r) fixing V.

Proof This follows immediately from Proposition 2.2.1. □

Proposition 3.1.1 implies that there is a category whose objects are bottomed Λ-algebras bound to V with morphisms bottomed homomorphisms fixing V. A bottomed Λ-algebra (X, p) is called *V-initial* if it is an initial object in this category, i.e., if it is bound to V and for each bottomed Λ-algebra (X', p') bound to V there exists a unique bottomed homomorphism $\pi : (X, p) \to (X', p')$ fixing V. If Λ is closed then 'initial' will be used instead of 'V-initial'. Thus in this special case a bottomed Λ-algebra (X, p) is initial if for each bottomed Λ-algebra (X', p') there exists a unique bottomed homomorphism $\pi : (X, p) \to (X', p')$.

Proposition 3.1.2 *There exists a V-initial bottomed Λ-algebra.*

Proof For the rest of the section let U be the B-family of sets with $U_{|A} = V$ and $U_b = \{\bot_b\}$ for all $b \in B \setminus A$; this B-family will be called the *$\bot$-trivial extension of V to B* (and note that U is considered just as a family of sets, and not as a family of bottomed sets). For each U-based Λ-algebra (X, p) there is then an associated bottomed Λ-algebra, also denoted by (X, p), obtained by stipulating that $\bot_b$ be the bottom element of X_b for each $b \in B$ (with $\bot_a$ the bottom element of V_a for each $a \in A$). Conversely, the underlying algebra of a bottomed Λ-algebra bound to V is a U-based Λ-algebra. For the existence of a V-initial bottomed Λ-algebra part of the following result is needed (the remainder being required for the proof of Propositions 3.1.3 and 3.1.4 below).

Lemma 3.1.1 *The bottomed Λ-algebra associated with a U-initial Λ-algebra is V-initial. Moreover, a bottomed Λ-algebra is V-initial if and only if the underlying algebra is U-initial.*

Proof This is essentially the same as the proof of Lemma 2.6.1. □

The existence of a V-initial bottomed Λ-algebra now follows from the first statement in Lemma 3.1.1 and from Proposition 2.5.2. □

For the present situation there is a result corresponding to Proposition 2.6.3 which gives an explicit characterisation of V-initial bottomed Λ-algebras. Before

preceeding, however, let us note that V-initial bottomed Λ-algebras are rather special. For example, the flat bottomed extension of an initial Λ-algebra is needed to describe the semantics of several programming languages, but it is almost never an initial bottomed Λ-algebra. In the following sections we will thus be considering a much larger class of bottomed algebras.

Let (X, p) be a bottomed Λ-algebra. A family of sets Y with $Y \subset X$ is said to be *invariant* in (X, p) if it is invariant in the underlying Λ-algebra, and it is *bottomed* if $\perp_b \in Y_b$ for each $b \in B$. A bottomed Λ-algebra (Y, q) is said to be a *bottomed subalgebra* of (X, p) if $Y \subset X$ as bottomed sets (i.e., $Y \subset X$ as sets with Y_b and X_b having the same bottom element for each $b \in B$) and q_k is the restriction of p_k to $Y^{k^\triangleright}$ for each $k \in K$. In this case the family of sets Y is clearly bottomed and invariant. Conversely, let Y be any invariant bottomed family and for each $k \in K$ let q_k denote the restriction of p_k to $Y^{k^\triangleright}$. Then (Y, q) is a bottomed subalgebra of (X, p), regarding Y_b as a bottomed set with bottom element $\perp_b$ (the bottom element of X_b) for each $b \in B$. If Y is an invariant bottomed family then the corresponding bottomed subalgebra (Y, q) is called the *bottomed subalgebra associated with* Y.

A bottomed Λ-algebra (X, p) bound to V is said to be *V-minimal* if X is the only invariant bottomed family $\check{X}$ in (X, p) such that $\check{X}_{|A} = V$. If Λ is closed then 'minimal' will be used instead of 'V-minimal'. Thus in this special case a bottomed Λ-algebra (X, p) is minimal X is the only invariant bottomed family in (X, p).

A bottomed Λ-algebra (X, p) is said to be *strictly regular* if the mapping p_k is injective for each $k \in K$ and for each $b \in B \setminus A$ the sets $\Im(p_k)$, $k \in K_b$, form a partition of $X_b^\natural$. Moreover, (X, p) is said to be *strictly unambiguous* if the mapping p_k is injective for each $k \in K$ and for each $b \in B$ the sets $\Im(p_k)$, $k \in K_b$, are disjoint subsets of $X_b^\natural$. The qualification 'strictly' is used here because we want to reserve 'regular' and 'unambiguous' for somewhat weaker properties to be introduced in Section 3.2. (As in Section 2.4, in the definition of being strictly unambiguous it would make no difference if B were replaced by $B \setminus A$, since $K_a = \varnothing$ for each $a \in A$.)

Proposition 3.1.3 *The following are equivalent for a bottomed Λ-algebra (X, p):*

(1) (X, p) is V-initial.
(2) (X, p) is V-minimal and strictly regular.
(3) (X, p) is V-minimal and strictly unambiguous.

Proof The following fact will be needed:

Lemma 3.1.2 *A bottomed Λ-algebra (X, p) bound to V is V-minimal if and only if its underlying algebra is U-minimal.*

Proof An invariant bottomed family $\breve{X}$ satisfies $\breve{X}_{|A} = V$ if and only if it contains U. This implies that (X,p) is V-minimal if and only if the underlying algebra is U-minimal. □

Let (X,p) be a bottomed Λ-algebra bound to V. Then by Lemma 3.1.2 (X,p) is V-minimal if and only if the underlying algebra is U-minimal. Furthermore, (since $U_b = \{\perp_b\}$ for each $b \in B \setminus A$) (X,p) is strictly regular if and only if the underlying algebra is U-regular and strictly unambiguous if and only if the underlying algebra is U-unambiguous. Proposition 3.1.3 therefore follows from Proposition 2.5.3 and the second statement in Lemma 3.1.1. □

The result corresponding to Proposition 2.6.5 also holds for bottomed algebras: A bottomed Λ-algebra (X,p) will be called *intrinsically free* if for each bottomed Λ-algebra (Y,q) and each A-family of bottomed mappings $\varrho : X_{|A} \to Y_{|A}$ there exists a unique bottomed homomorphism $\pi : (X,p) \to (Y,q)$ such that $\pi_{|A} = \varrho$.

Proposition 3.1.4 *A bottomed Λ-algebra bound to V is V-initial if and only if it is intrinsically free.*

Proof An intrinsically free bottomed Λ-algebra bound to V is clearly V-initial and the converse follows immediately from Lemma 2.5.4 and Lemma 3.1.1. □

Lemma 3.1.3 *Let π, ϱ be bottomed homomorphisms from a V-minimal bottomed Λ-algebra (X,p) to a bottomed Λ-algebra (Y,q) with $\pi_{|A} = \varrho_{|A}$. Then $\pi = \varrho$. In particular, if (Y,q) is bound to V then there exists at most one bottomed homomorphism from (X,p) to (Y,q) fixing V.*

Proof This follows from Proposition 2.3.3 (1) and Lemma 3.1.2. □

In Section 2.3 a sufficient condition for a Λ-algebra to be minimal was given in terms of a grading. For a bottomed Λ-algebra (X,p) the definition of a grading has to be relaxed a bit: A B-family of mappings $\#$ with $\#_b : X_b \to \mathbb{N}$ for each $b \in B$ will be called a *bottomed grading* for (X,p) if $\#_b(\perp_b) = 0$ for all $b \in B$ and $\#_{k^\triangleright\eta}(v(\eta)) < \#_{k_\triangleleft}(p_k(v))$ for all $\eta \in \langle k^\triangleright \rangle$ whenever $v \in X^{k^\triangleright}$ is such that $p_k(v) \neq \perp_{k_\triangleleft}$. If there exists a bottomed grading then (X,p) is said to be *graded*.

Lemma 3.1.4 *If (X,p) is a graded bottomed Λ-algebra bound to V and*

$$X_b = \{\perp_b\} \cup \bigcup_{k \in K_b} \Im(p_k)$$

for each $b \in B \setminus A$ then (X,p) is V-minimal.

Proof Let $\#$ be a bottomed grading for (X, p) and let $\hat{X}$ be the minimal invariant bottomed family with $\hat{X}_{|A} = V$; suppose $\hat{X} \neq X$. There thus exists $b \in B \setminus A$ and $x \in X_b \setminus \hat{X}_b$ such that $\#_b(x) \leq \#_{b'}(x')$ whenever $x' \in X_{b'} \setminus \hat{X}_{b'}$ for some $b' \in B$. In particular, $x \neq \bot_b$, since $\bot_b \in \hat{X}_b$, and hence $x \in \Im(p_k)$ for some $k \in K_b$. There therefore exists $v \in X^{k^\triangleright}$ with $x = p_k(v)$. But it then follows that $\#_{k^\triangleright\eta}(v(\eta)) < \#_{k_\triangleleft}(x)$ and hence that $v(\eta) \in \hat{X}_{k^\triangleright\eta}$ for each $\eta \in \langle k^\triangleright \rangle$ (by the minimality of $\#_b(x)$). However, this implies $x \in \hat{X}_b$, since the family $\hat{X}$ is invariant, which is a contradiction. $\square$

Proposition 3.1.5 *If (X, p) is a V-minimal bottomed Λ-algebra then*

$$X_b = \{\bot_b\} \cup \bigcup_{k \in K_b} \Im(p_k)$$

for each $b \in B \setminus A$. Moreover, a graded bottomed Λ-algebra (X, p) bound to V is V-minimal if and only if this equality holds for each $b \in B \setminus A$.

Proof This follows from Lemmas 3.1.2 and 3.1.4 and Proposition 2.3.1. $\square$

For the rest of the section consider the case when Λ is the disjoint union of the signatures Λ_i, $i \in F$ (as defined at the end of Section 2.2). For each $i \in F$ let (X^i, p^i) be a bottomed Λ_i-algebra. Then the sum $\oplus_{i \in F}(X^i, p^i)$ is clearly a bottomed Λ-algebra. Moreover, if A_i is the parameter set of Λ_i and (X^i, p^i) is bound to the A_i-family of bottomed sets V^i for each $i \in F$ then $\oplus_{i \in F}(X^i, p^i)$ is bound to the A-family V with $V_a = V^i_a$ for each $a \in A^i_a$.

Proposition 3.1.6 *If (X^i, p^i) is V^i-minimal for each $i \in F$ then $\oplus_{i \in F}(X^i, p^i)$ is V-minimal.*

Proof Straightforward. $\square$

Example 3.1.2 The following notation will be employed here (and also later): If Z is a set then $\mathsf{bot}\,(Z)$ denotes a disjoint copy of Z; the element in $\mathsf{bot}\,(Z)$ corresponding to the element $z \in Z$ will be denoted by $z^\perp$ (so $\mathsf{bot}\,(Z) = \{z^\perp : z \in Z\}$).

Let Λ be the signature in Example 2.2.1 and consider the bottomed Λ-algebra (Y, q) defined by

$Y_{\mathtt{bool}} = \mathbb{B}^\perp = \mathbb{B} \cup \{\perp_{\mathtt{bool}}\}$,
$Y_{\mathtt{nat}} = \mathbb{N} \cup \mathsf{bot}\,(\mathbb{N})$ with $0^\perp = \perp_{\mathtt{nat}}$,
$Y_{\mathtt{int}} = \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}$, $\quad Y_{\mathtt{pair}} = Y_{\mathtt{int}}^2 \cup \{\perp_{\mathtt{pair}}\}$,
$Y_{\mathtt{list}} = Y_{\mathtt{int}}^* \cup \mathsf{bot}\,(Y_{\mathtt{int}}^*)$ with $\varepsilon^\perp = \perp_{\mathtt{list}}$,

all these unions being considered to be disjoint and Y_b being considered as a bottomed set with bottom element $\perp_b$ for each $b \in B$,

$q_{\mathtt{True}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{True}}(\varepsilon) = \mathrm{T}$,
$q_{\mathtt{False}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{False}}(\varepsilon) = \mathrm{F}$,
$q_{\mathtt{Zero}} : \mathbb{I} \to Y_{\mathtt{nat}}$ with $q_{\mathtt{Zero}}(\varepsilon) = 0$,
$q_{\mathtt{Succ}} : Y_{\mathtt{nat}} \to Y_{\mathtt{nat}}$ with

$$q_{\mathtt{Succ}}(x) = \begin{cases} n+1 & \text{if } x = n \text{ for some } n \in \mathbb{N}, \\ (n+1)^\perp & \text{if } x = n^\perp \text{ for some } n \in \mathbb{N}, \end{cases}$$

$q_{\underline{n}} : \mathbb{I} \to Y_{\mathtt{int}}$ with $q_{\underline{n}}(\varepsilon) = n$ for each $n \in \mathbb{Z}$,
$q_{\mathtt{Pair}} : Y_{\mathtt{int}} \times Y_{\mathtt{int}} \to Y_{\mathtt{pair}}$ with $q_{\mathtt{Pair}}(x_1, x_2) = (x_1, x_2)$,
$q_{\mathtt{Nil}} : \mathbb{I} \to Y_{\mathtt{list}}$ with $q_{\mathtt{Nil}}(\varepsilon) = \varepsilon$,
$q_{\mathtt{Cons}} : Y_{\mathtt{int}} \times Y_{\mathtt{list}} \to Y_{\mathtt{list}}$ with

$$q_{\mathtt{Cons}}(x, z) = \begin{cases} x \triangleleft s & \text{if } z = s \text{ for some } s \in Y_{\mathtt{int}}^*, \\ (x \triangleleft s)^\perp & \text{if } z = s^\perp \text{ for some } s \in Y_{\mathtt{int}}^*. \end{cases}$$

The reader is left to check that (Y, q) is an initial bottomed Λ-algebra. Of course, the underlying Λ-algebra is an extension of the Λ-algebra (X, p) introduced in Example 2.2.1.

Note that an element of $Y_{\mathtt{list}}$ has either the form $x_1 \cdots x_n$ or the form $(x_1 \cdots x_n)^\perp$, where $n \geq 0$ and $x_j \in \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}$ for $j = 1, \ldots, n$. The element $x_1 \cdots x_n$ describes a ‘real’ list with n components (although some or all of these components may be ‘undefined’). The element $(x_1 \cdots x_n)^\perp$, on the other hand, should be thought of as a ‘partial’ list containing at least n components, of which the first n components are ‘known’ to be $x_1, \ldots, x_n$.

Example 3.1.3 Again let Λ be the signature in Example 2.2.3 and let V be an A-family of bottomed sets. Then a bottomed Λ-algebra (Y, q) bound to V can be defined by letting

$Y_{\mathtt{bool}} = \mathbb{B}^{\perp} = \mathbb{B} \cup \{\perp_{\mathtt{bool}}\}$, $\;Y_{\mathtt{atom}} = \mathbb{I}^{\perp} = \mathbb{I} \cup \{\perp_{\mathtt{atom}}\}$,
$Y_{\mathtt{int}} = \mathbb{Z}^{\perp} = \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}$, $\;Y_{\mathtt{pair}} = (V_{\mathtt{x}} \times V_{\mathtt{y}}) \cup \{\perp_{\mathtt{pair}}\}$,
$Y_{\mathtt{list}} = V_{\mathtt{z}}^* \cup \mathsf{bot}\,(V_{\mathtt{z}}^*)$ with $\perp_{\mathtt{list}} = \varepsilon^{\perp}$.
$Y_{\mathtt{lp}} = Y_{\mathtt{pair}} \cup Y_{\mathtt{list}} \cup \{\perp_{\mathtt{lp}}\}$,
$Y_{\mathtt{x}} = V_{\mathtt{x}},\; Y_{\mathtt{y}} = V_{\mathtt{y}},\; Y_{\mathtt{z}} = V_{\mathtt{z}}$,

all these unions being considered to be disjoint and Y_b being considered as a bottomed set with bottom element $\perp_b$ for each $b \in B \setminus A$,

$q_{\mathtt{True}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{True}}(\varepsilon) = \mathrm{T}$,
$q_{\mathtt{False}} : \mathbb{I} \to Y_{\mathtt{bool}}$ with $q_{\mathtt{False}}(\varepsilon) = \mathrm{F}$,
$q_{\mathtt{Atom}} : \mathbb{I} \to Y_{\mathtt{atom}}$ with $q_{\mathtt{Atom}}(\varepsilon) = \varepsilon$,
$q_{\underline{n}} : \mathbb{I} \to Y_{\mathtt{int}}$ with $q_{\underline{n}}(\varepsilon) = n$ for each $n \in \mathbb{Z}$,
$q_{\mathtt{Pair}} : Y_{\mathtt{x}} \times Y_{\mathtt{y}} \to Y_{\mathtt{pair}}$ with $q_{\mathtt{Pair}}(x, y) = (x, y)$,
$q_{\mathtt{Nil}} : \mathbb{I} \to Y_{\mathtt{list}}$ with $q_{\mathtt{Nil}}(\varepsilon) = \varepsilon$,
$q_{\mathtt{Cons}} : Y_{\mathtt{z}} \times Y_{\mathtt{list}} \to Y_{\mathtt{list}}$ with

$$q_{\mathtt{Cons}}(x, z) = \begin{cases} x \triangleleft s & \text{if } z = s \text{ for some } s \in V_{\mathtt{z}}^*, \\ (x \triangleleft s)^{\perp} & \text{if } z = s^{\perp} \text{ for some } s \in V_{\mathtt{z}}^*. \end{cases}$$

$q_{\mathtt{L}} : Y_{\mathtt{list}} \to Y_{\mathtt{lp}}$ with $q_{\mathtt{L}}(s) = s$,
$q_{\mathtt{P}} : Y_{\mathtt{pair}} \to Y_{\mathtt{lp}}$ with $q_{\mathtt{P}}(p) = p$.

The reader is left to show, making use of Proposition 3.1.3, that (Y, q) is a V-initial bottomed Λ-algebra.

3.2 Regular bottomed algebras

A bottomed Λ-algebra (X, p) is said to be *regular* if for each $b \in B \setminus A$ and each $x \in X_b^{\natural}$ there exists a unique $k \in K_b$ and a unique $v \in X^{k^{\triangleright}}$ such that $p_k(v) = x$. Regularity is essential if 'case' or 'pattern matching' operators (as they occur in all modern functional programming languages) are to be defined. It says that if $b \in B \setminus A$ and $x \in X_b$ is not completely undefined then x can be constructed in a unique way by applying a constructor p_k (with $k \in K_b$) to one of its arguments.

Lemma 3.2.1 *A strictly regular bottomed Λ-algebra is regular. Conversely, a regular bottomed Λ-algebra (X, p) is strictly regular if and only if $\perp_{k_{\triangleleft}} \notin \Im(p_k)$ for each $k \in K$.*

Proof This is clear. □

By Proposition 3.1.3 each V-initial bottomed Λ-algebra is strictly regular and hence by Lemma 3.2.1 it is regular. Moreover, if (Y,q) is an initial Λ-algebra then the flat bottomed extension $(Y^\perp, q^\perp)$ of (Y,q) is a regular bottomed Λ-algebra.

Let us now look at the problem left over from the previous section of knowing when a bottomed Λ-algebra (X,p) is a bottomed extension of a suitable Λ-algebra (Y,q) describing the defined data objects. In practice it can be assumed that (Y,q) is U-initial for some A-family of sets U and then (Y,q) must be the unique U-minimal subalgebra of the underlying algebra (X,p). Note that here $U \subset X^\natural_{|A}$, since $\perp_a \notin Y_a$ for each $a \in A$.

A simple condition which, together with being regular, solves this problem is the following: A bottomed Λ-algebra (X,p) will be called $\natural$-*invariant* if the family $X^\natural$ is invariant in the underlying algebra, i.e., if $p_k(v) \neq \perp_{k_\triangleleft}$ whenever $v \in X^{k^\triangleright}$ is such that $v(\eta) \neq \perp_{k^\triangleright \eta}$ for all $\eta \in \langle k^\triangleright \rangle$. In particular, a strictly regular bottomed Λ-algebra as well as any flat bottomed extension is $\natural$-invariant.

Proposition 3.2.1 *Let (X,p) be a regular $\natural$-invariant bottomed Λ-algebra and let U be an A-family of sets with $U \subset X^\natural_{|A}$. Then the unique U-minimal subalgebra (Y,q) of the underlying algebra (X,p) is U-initial and $\perp_b \notin Y_b$ for each $b \in B$ (and so in particular (X,p) is a bottomed extension of (Y,q)).*

Proof Let $\breve{U}$ be the trivial extension of U to B (i.e., the B-family of sets with $\breve{U}_{|A} = U$ and $\breve{U}_{|B\setminus A} = \varnothing$). The proof of Proposition 2.6.6 shows that Y is the minimal invariant family in the underlying Λ-algebra (X,p) containing $\breve{U}$. But $X^\natural$ is also an invariant family in (X,p) containing $\breve{U}$, and therefore $Y \subset X^\natural$, i.e., $\perp_b \notin Y_b$ for each $b \in B$. Now, since (X,p) is regular and $Y \subset X^\natural$, it follows that q_k is injective for each $k \in K$ and that the sets $\Im(q_k)$, $k \in K_b$, are disjoint subsets of Y_b for each $b \in B \setminus A$. Hence (Y,q) is unambiguous, and so by Proposition 2.6.3 it is U-initial. □

Note the following special case of Proposition 3.2.1: If the signature Λ is closed (i.e., if $A = \varnothing$) and (X,p) is a regular $\natural$-invariant bottomed Λ-algebra then the unique minimal subalgebra (Y,q) of the underlying algebra (X,p) is initial and $\perp_b \notin Y_b$ for each $b \in B$ (and so in particular (X,p) is a bottomed extension of (Y,q)).

Recall that if X and Y are bottomed sets then a bottomed mapping $f : X \to Y$ is said to be *proper* if $f(X^\natural) \subset Y^\natural$ (i.e., if $f(x) \neq \perp_Y$ for all $x \in X \setminus \{\perp_X\}$). A bottomed homomorphism is called *proper* if it is a family of proper mappings.

Lemma 3.2.2 *Let (X,p), (Y,q) be bottomed Λ-algebras with (Y,q) $\natural$-invariant and suppose there exists a proper bottomed homomorphism $\pi : (X,p) \to (Y,q)$. Then (X,p) is also $\natural$-invariant.*

Proof This follows from Lemma 2.3.1 (2), since $X_b^\natural = \pi_b^{-1}(Y_b^\natural)$ for each $b \in B$. □

The rest of the section is taken up with the statement and proof of a somewhat technical result which, however, plays a crucial role in the next section. In what follows let V be an A-family of bottomed sets, let $(H, \diamond)$ be a bottomed Λ-algebra and let $\sigma : V \to H_{|A}$ be an A-family of proper bottomed mappings. A bottomed Λ-algebra (X, p) bound to V is said to be *classified by* $(H, \diamond)$ *and* σ if there exists a proper bottomed homomorphism $\pi : (X, p) \to (H, \diamond)$ with $\pi_{|A} = \sigma$.

Proposition 3.2.2 *There exists a V-minimal regular bottomed Λ-algebra which is classified by $(H, \diamond)$ and σ. Moreover, any such bottomed Λ-algebra (X, p) is an initial object: For each bottomed Λ-algebra (Y, q) bound to V and classified by $(H, \diamond)$ and σ there exists a unique bottomed homomorphism $\pi : (X, p) \to (Y, q)$ fixing V.*

Proof This occupies the rest of the section. □

It is useful to define a bottomed Λ-algebra (X, p) to be *unambiguous* if for each $b \in B$ and each $x \in X_b^\natural$ there exists at most one $k \in K_b$ with $x \in \Im(p_k)$ and, moreover, if there is such a k then there exists at most one $v \in X^{k^\triangleright}$ with $x = p_k(v)$. In particular, any regular bottomed Λ-algebra is unambiguous.

Lemma 3.2.3 *A V-minimal bottomed Λ-algebra is unambiguous if and only if it is regular.*

Proof This follows from the first statement in Proposition 3.1.5. □

Lemma 3.2.4 *There exists a V-minimal regular bottomed Λ-algebra which is classified by $(H, \diamond)$ and σ.*

Proof By Proposition 3.1.2 there exists a V-initial bottomed Λ-algebra (Z, r) and by Proposition 3.1.4 there then exists a unique bottomed homomorphism δ from (Z, r) to $(H, \diamond)$ such that $\delta_{|A} = \sigma$. Put

$$Z'_b = \{z \in Z_b : \delta_b(z) \neq \perp_b\} \cup \{\perp_b\}$$

for each $b \in B$; hence Z' is a B-family of bottomed sets with $Z'_{|A} = V$. For each $k \in K$ define a mapping $r'_k : Z^{k^\triangleright} \to Z_{k_\triangleleft}$ by letting

$$r'_k(v) = \begin{cases} r_k(v) & \text{if } r_k(v) \in Z'_{k_\triangleleft}, \\ \perp_{k_\triangleleft} & \text{otherwise.} \end{cases}$$

Then (Z, r') is a bottomed Λ-algebra bound to V, and it is easy to see that (Z, r') is unambiguous (since by Proposition 3.1.3 (Z, r) is unambiguous and if $z \in Z_b^\natural$ with $z = r'_k(v)$ then also $z = r_k(v)$). Moreover, Z' is an invariant bottomed family in (Z, r'), since $\Im(r'_k) \subset Z'_{k_\triangleleft}$ for each $k \in K$.

Let X be the minimal invariant bottomed family in (Z, r') with $X_{|A} = V$, and let (X, p) be the associated bottomed subalgebra (so $p_k : X^{k^\triangleright} \to X_{k_\triangleleft}$ is the restriction of r'_k to $X^{k^\triangleright}$). Then (X, p) is a bottomed Λ-algebra bound to V, and by construction it is V-minimal. But (X, p), being a bottomed subalgebra of the unambiguous bottomed Λ-algebra (Z, r'), is itself unambiguous, and therefore by Lemma 3.2.3 (X, p) is regular, i.e., (X, p) is a V-minimal regular bottomed Λ-algebra. Note that $X \subset Z'$, since Z' is an invariant bottomed family in (Z, r') with $Z'_{|A} = V$.

For each $b \in B$ let $\pi_b : X_b \to H_b$ be the restriction of δ_b to X_b. Then π_b is a proper bottomed mapping, since if $x \in X_b^\natural$ then $x \in Z'_b \setminus \{\perp_b\}$ and so $\pi_b(x) = \delta_b(x) \neq \perp_b$. Moreover, $\pi_{|A} = \sigma$. The proof will therefore be completed by showing that π is a homomorphism from (X, p) to $(H, \diamond)$. Thus consider $k \in K$ and $v \in X^{k^\triangleright}$. If $r_k(v) \in Z'_{k_\triangleleft}$ then $r'_k(v) = r_k(v)$ and in this case

$$\pi_{k_\triangleleft}(p_k(v)) = \delta_{k_\triangleleft}(r'_k(v)) = \delta_{k_\triangleleft}(r_k(v)) = \diamond_k(\delta^{k^\triangleright}(v)) = \diamond_k(\pi^{k^\triangleright}(v)) \; .$$

On the other hand, if $r_k(v) \notin Z'_{k_\triangleleft}$ then $\delta_{k_\triangleleft}(r_k(v)) = \perp_{k_\triangleleft}$ and $r'_k(v) = \perp_{k_\triangleleft}$, and hence also $\delta_{k_\triangleleft}(r'_k(v)) = \perp_{k_\triangleleft}$, and so it again follows that

$$\pi_{k_\triangleleft}(p_k(v)) = \delta_{k_\triangleleft}(r'_k(v)) = \perp_{k_\triangleleft} = \delta_{k_\triangleleft}(r_k(v)) = \diamond_k(\delta^{k^\triangleright}(v)) = \diamond_k(\pi^{k^\triangleright}(v)) \; .$$

This shows that $\pi_{k_\triangleleft}(p_k(v)) = \diamond_k(\pi^{k^\triangleright}(v))$ for all $v \in X^{k^\triangleright}$, $k \in K$, i.e., π is a homomorphism from (X, p) to $(H, \diamond)$. $\square$

For the moment let us just assume that (X, p) is a V-minimal regular bottomed Λ-algebra.

Lemma 3.2.5 *There exists a unique B-family of mappings $\#$ with $\#_b : X_b \to \mathbb{N}$ for each $b \in B$ such that $\#_a(x) = 0$ for each $x \in X_a$, $a \in A$, $\#_b(\perp_b) = 0$ for each $b \in B$, and for each $k \in K$*

$$\#_{k^\triangleright}(p_k(v)) = 1 + \max\{\#_{k^\triangleright\eta}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\}$$

for all $v \in X^{k^\triangleright}$ with $p_k(v) \neq \perp_{k_\triangleleft}$ (the maximum being taken to be 0 if $k^\triangleright = \varepsilon$).

Proof This is very similar to the proof of Lemma 2.4.5. We give the proof in full, however, because Lemma 3.2.5 is the key to the proof of Proposition 3.2.2. The family $\#$ will be obtained as the limit of a sequence $\{\#^m\}_{m \geq 0}$, where $\#^m$ is a B-family of mappings with $\#^m_b : X_b \to \mathbb{N}$ for each $b \in B$. First define

$\#_b^0 = 0$ for each $b \in B$. Next suppose the family $\#^m$ has already been defined for some $m \in \mathbb{N}$. Then, since (X, p) is a regular bottomed Λ-algebra, there exists a unique family of mappings $\#^{m+1}$ such that $\#_a^{m+1}(x) = 0$ for each $x \in X_a$, $a \in A$, $\#_b^{m+1}(\perp_b) = 0$ for each $b \in B$, and for each $k \in K$

$$\#_{k^\triangleright}^{m+1}(p_k(v)) = 1 + \max\{\#_{k^\triangleright\eta}^m(v(\eta)) : \eta \in \langle k^\triangleright \rangle\}$$

for all $v \in X^{k^\triangleright}$ with $p_k(v) \neq \perp_{k_\triangleleft}$ (the maximum being taken to be 0 if $k^\triangleright = \varepsilon$). By induction this defines the family $\#^m$ for each $m \in \mathbb{N}$.

Now $\#^m \leq \#^{m+1}$ holds for each $m \in \mathbb{N}$: By definition $\#_a^m(x) = \#_a^{m+1}(x) = 0$ for each $x \in X_a$, $a \in A$, and $\#_b^m(\perp_b) = \#_b^{m+1}(\perp_b) = 0$ for each $b \in B$; also $\#^0 \leq \#^1$ holds by definition. But if $\#^m \leq \#^{m+1}$ for some $m \in \mathbb{N}$ and $k \in K$ then

$$\begin{aligned} \#_{k_\triangleleft}^{m+1}(p_k(v)) &= 1 + \max\{\#_{k^\triangleright\eta}^m(v(\eta)) : \eta \in \langle k^\triangleright \rangle\} \\ &\leq 1 + \max\{\#_{k^\triangleright\eta}^{m+1}(v(\eta)) : \eta \in \langle k^\triangleright \rangle\} = \#_{k_\triangleleft}^{m+2}(p_k(v)) \end{aligned}$$

for all $v \in X^{k^\triangleright}$ with $p_k(v) \neq \perp_{k_\triangleleft}$. This implies that $\#_b^{m+1}(x) \leq \#_b^{m+2}(x)$ for all $x \in X_b^\natural$, $b \in B \setminus A$, and hence that $\#^{m+1} \leq \#^{m+2}$. Thus by induction $\#^m \leq \#^{m+1}$ for each $m \in \mathbb{N}$.

Moreover, the sequence $\{\#_b^m(x)\}_{m \geq 0}$ is bounded for each $x \in X_b$, $b \in B$: Let X'_b denote the set of those elements $x \in X_b$ for which this is the case. Then X' is bottomed, $X'_{|A} = V$, and it is easily checked that the B-family X' is invariant, and hence $X' = X$, since (X, p) is V-minimal.

Let $x \in X_b$; then by the above $\{\#_b^m(x)\}_{m \geq 0}$ is a bounded increasing sequence from $\mathbb{N}$, and so there exists an element $\#_b(x) \in \mathbb{N}$ such that $\#_b^m(x) = \#_b(x)$ for all but finitely many m. This defines a mapping $\#_b : X_b \to \mathbb{N}$ for each $b \in B$, and it immediately follows that the family $\#$ has the required property. It remains to show the uniqueness, so suppose $\#'$ is another B-family of mappings with this property. For each $b \in B$ let $X'_b = \{x \in X_b : \#'_b(x) = \#_b(x)\}$; then X' is clearly an invariant bottomed family with $X'_{|A} = V$, and hence $X' = X$, since (X, p) is V-minimal. □

In what follows again let (Z, r) be a fixed V-initial bottomed Λ-algebra, and let λ be the unique bottomed homomorphism from (Z, r) to (X, p) fixing V.

Lemma 3.2.6 *There exists a unique family of bottomed mappings $\varrho : X \to Z$ fixing V and such that for each $k \in K$*

$$\varrho_{k_\triangleleft}(p_k(v)) = r_k(\varrho^{k^\triangleright}(v))$$

for all $v \in X^{k^\triangleright}$ with $p_k(v) \neq \perp_{k_\triangleleft}$. Moreover, $\lambda_b(\varrho_b(x)) = x$ for all $x \in X_b$, $b \in B$.

Proof Let $\#$ be the family of mappings given by Lemma 3.2.5 and for each $b \in B$, $m \in \mathbb{N}$ let $X_b^m = \{x \in X_b : \#_b(x) = m\}$. Define ϱ_b on X_b^m for each $b \in B$ using induction on m. If $x \in X_b^0$ then either $b \in A$ or $x = \perp_b$, in which case put $\varrho_b(x) = x$. Now let $m > 0$ and suppose ϱ_c is already defined on X_c^k for each $c \in B$ and for all $k < m$. Let $x \in X_b^m$; then in particular $b \in B \setminus A$ and $x \in X_b^\natural$ and so there exists a unique $k \in K_b$ and a unique element $v \in X^{k^\triangleright}$ such that $x = p_k(v)$. Moreover, $v(\eta) \in X_{k^\triangleright\eta}^{m_\eta}$ for some m_η with $m_\eta < m$, which means that $\varrho_{k^\triangleright\eta}(v(\eta))$ is already defined for each $\eta \in \langle k^\triangleright \rangle$. It thus makes sense to put $\varrho_b(x) = r_b(\varrho^{k^\triangleright}(v))$ (where of course $\varrho^{k^\triangleright}(v)$ is the element $v' \in Z^{k^\triangleright}$ with $v'(\eta) = \varrho_{k^\triangleright\eta}(v(\eta))$ for each $\eta \in \langle k^\triangleright \rangle$). In this way ϱ_b is defined on X_b^m for each $m \in \mathbb{N}$ and the family ϱ has the required property by construction. The uniqueness of the family ϱ also follows using induction on m.

Finally, for each $b \in B$ let $X'_b = \{x \in X_b : \lambda_b(\varrho_b(x)) = x\}$; then $X'_{|A} = V$ and the family X' is bottomed. Moreover, it is also invariant. (Let $k \in K$ and $v \in X^{k^\triangleright}$ with $v(\eta) \in X'_{k^\triangleright\eta}$ for each $\eta \in \langle k^\triangleright \rangle$; put $x = p_k(v)$. If $x = \perp_{k_\triangleleft}$ then $x \in X'_{k_\triangleleft}$ holds immediately. On the other hand, if $x \in X_{k_\triangleleft}^\natural$ then $p_k(v) \neq \perp_{k_\triangleleft}$; in this case $\varrho_{k_\triangleleft}(x) = r_k(\varrho^{k^\triangleright}(v))$ and hence, since $\lambda_{k^\triangleright\eta}(\varrho_\eta(v(\eta))) = v(\eta)$ for each $\eta \in \langle k^\triangleright \rangle$ and so by Lemma 2.1.3 (2) $\lambda^{k^\triangleright}(\varrho^{k^\triangleright}(v)) = v$, it follows that

$$\lambda_{k_\triangleleft}(\varrho_{k_\triangleleft}(x)) = \lambda_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v))) = p_k(\lambda^{k^\triangleright}(\varrho^{k^\triangleright}(v))) = p_k(v) = x\ .$$

Thus in both cases $x \in X'_{k_\triangleleft}$.) But (X, p) is V-minimal and thus $X' = X$, i.e., $\lambda_b(\varrho_b(x)) = x$ for all $x \in X_b$, $b \in B$. $\square$

Now let (Y, q) be any bottomed Λ-algebra and let $\tau : V \to Y_{|A}$ be any family of bottomed mappings. By Proposition 3.1.4 there exists a unique bottomed homomorphism $\mu : (Z, r) \to (Y, q)$ such that $\mu_{|A} = \tau$. Let M, $N \subset Z$ be the families defined by $M_b = \{z \in Z_b : \lambda_b(z) = \perp_b\}$ and $N_b = \{z \in Z_b : \mu_b(z) = \perp_b\}$ for each $b \in B$.

Lemma 3.2.7 *Let $\varrho : X \to Z$ be the family defined in Lemma 3.2.6. If $M \subset N$ then $\omega = \mu \circ \varrho$ is a bottomed homomorphism from (X, p) to (Y, q) with $\omega_{|A} = \tau$. Moreover, ω is the unique such bottomed homomorphism.*

Proof Let $k \in K$ and $v \in X^{k^\triangleright}$; then by Lemma 2.1.3 (2)

$$q_k(\omega^{k^\triangleright}(v)) = q_k(\mu^{k^\triangleright}(\varrho^{k^\triangleright}(v))) = \mu_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v)))\ .$$

If $p_k(v) \neq \perp_{k_\triangleleft}$ then $\omega_{k_\triangleleft}(p_k(v)) = \mu_{k_\triangleleft}(\varrho_{k_\triangleleft}(p_k(v))) = \mu_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v)))$, and so in this case $\omega_{k_\triangleleft}(p_k(v)) = q_k(\omega^{k^\triangleright}(v))$. But if $p_k(v) = \perp_{k_\triangleleft}$ then $\lambda_{k^\triangleright\eta}(\varrho_{k^\triangleright\eta}(v(\eta))) = v(\eta)$ for each $\eta \in \langle k^\triangleright \rangle$ and so by Lemma 2.1.3 (2) $\lambda^{k^\triangleright}(\varrho^{k^\triangleright}(v)) = v$, thus

$$\lambda_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v))) = p_k(\lambda^{k^\triangleright}(\varrho^{k^\triangleright}(v))) = p_k(v) = \perp_{k_\triangleleft}\ ,$$

which means that $r_k(\varrho^{k^\triangleright}(v)) \in M_{k_\triangleleft}$. Therefore $\mu_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v))) = \perp_{k_\triangleleft}$, since $M_{k_\triangleleft} \subset N_{k_\triangleleft}$, and again

$$q_k(\omega^{k^\triangleright}(v)) = \mu_{k_\triangleleft}(r_k(\varrho^{k^\triangleright}(v))) = \perp_{k_\triangleleft} = \omega_{k_\triangleleft}(\perp_{k_\triangleleft}) = \omega_{k_\triangleleft}(p_k(v)) \ .$$

This shows that ω is a homomorphism, and it is clear that ω is bottomed and that $\omega_{|A} = \tau$. The uniqueness follows from Lemma 3.1.3. $\square$

Suppose now that (X, p) is a V-minimal regular bottomed Λ-algebra classified by $(H, \diamond)$ and σ and that (Y, q) is any bottomed Λ-algebra bound to V and classified by $(H, \diamond)$ and σ. Suppose further that $\tau = \mathrm{id}$, and so $\mu : (Z, r) \to (Y, q)$ is the unique bottomed homomorphism fixing V.

Lemma 3.2.8 *With the above assumptions $M = N$.*

Proof Again let $\delta : (Z, r) \to (H, \diamond)$ be the unique bottomed homomorphism such that $\delta_{|A} = \sigma$ and let $L \subset Z$ be the family with $L_b = \{z \in Z_b : \delta_b(z) = \perp_b\}$ for each $b \in B$. Now by definition there exists a proper bottomed homomorphism $\pi : (X, p) \to (H, \diamond)$ such that $\pi_{|A} = \sigma$, and by Proposition 3.1.1 (2) $\pi \circ \lambda$ is then a bottomed homomorphism from (Z, r) to $(H, \diamond)$ with $\pi_a(\lambda_a(z)) = \pi_a(z) = \sigma_a(z)$ for each $z \in Z_a$, $a \in A$. But δ is the unique such homomorphism and hence $\delta = \pi \circ \lambda$. In particular, $M = L$, since π is a family of proper bottomed mappings. In the same way $N = L$ and therefore $M = N$. $\square$

Proof of Proposition 3.2.2 This follows from Lemmas 3.2.4, 3.2.7 and 3.2.8. $\square$

To end the section consider the case when Λ is the disjoint union of the signatures Λ_i, $i \in F$ (as defined at the end of Section 2.2).

Proposition 3.2.3 *For each $i \in F$ let (X^i, p^i) be a regular bottomed Λ_i-algebra. Then the sum $\oplus_{i \in F}(X^i, p^i)$ is regular.*

Proof Straightforward. $\square$

3.3 Algebras associated with a head type

In this section we introduce the concept of a head type, which is a simple kind of bottomed Λ-algebra. Head types will be used to classify the bottomed Λ-algebras which typically arise when dealing with functional programming languages. The main result is Proposition 3.3.1; this is just a special case of Proposition 3.2.2. In particular, the set-up considered includes as special cases flat bottomed extensions and V-initial bottomed algebras.

Let $\perp$ and $\natural$ be distinct elements and regard $\mathbb{T} = \{\perp, \natural\}$ as a bottomed set with bottom element $\perp$. A bottomed Λ-algebra $(H, \diamond)$ will be called a *head type* if $H_a = \mathbb{T}$ for each $a \in A$. Note that if V is any A-family of bottomed sets and $(H, \diamond)$ is a head type then there is a unique A-family of proper bottomed mappings $\sigma : V \to H_{|A}$.

If $(H, \diamond)$ is a head type then a bottomed Λ-algebra (X, p) is said to be a bottomed Λ-algebra *associated with* $(H, \diamond)$ or, more simply, to be an $(H, \diamond)$-*algebra* if there exists a proper bottomed homomorphism from (X, p) to $(H, \diamond)$.

In the context of programming languages the choice of a head type can be seen as a design decision for the language being considered. We will usually assume that a head type $(H, \diamond)$ is given and then only be interested in $(H, \diamond)$-algebras.

Proposition 3.3.1 *Let $(H, \diamond)$ be a head type and V be an A-family of bottomed sets. Then there exists a V-minimal regular $(H, \diamond)$-algebra. Moreover, any such $(H, \diamond)$-algebra is an initial object in the full subcategory of $(H, \diamond)$-algebras bound to V.*

Proof This is just a special case of Proposition 3.2.2, since a bottomed Λ-algebra bound to V is an $(H, \diamond)$-algebra if and only if it is classified by $(H, \diamond)$ and the unique A-family of proper bottomed mappings $\sigma : V \to H_{|A}$. $\square$

Lemma 3.3.1 *A bottomed subalgebra (Y, q) of an $(H, \diamond)$-algebra (X, p) is itself an $(H, \diamond)$-algebra.*

Proof If $\pi : (X, p) \to (H, \diamond)$ is a proper bottomed homomorphism and ϱ_b is the restriction of π_b to Y_b for each $b \in B$ then ϱ is a proper bottomed homomorphism from (Y, q) to $(H, \diamond)$. $\square$

Proposition 3.3.2 *A bottomed Λ-algebra which is an $(H, \diamond)$-algebra for some $\natural$-invariant head type $(H, \diamond)$ is itself $\natural$-invariant.*

Proof This follows immediately from Lemma 3.2.2. $\square$

A head type $(H, \diamond)$ will be called *simple* if $H_b = \mathbb{T}$ for each $b \in B$. Let $(H, \diamond)$ be a simple head type and (X, p) be a bottomed Λ-algebra. Then there is only one A-family of proper bottomed mappings $\varepsilon : X \to H$, namely with $\varepsilon_b(\bot_b) = \bot$ and $\varepsilon_b(x) = \natural$ for all $x \in X_b^\natural$, $b \in B$. Thus (X, p) is an $(H, \diamond)$-algebra if and only if this family ε is a homomorphism from (X, p) to $(H, \diamond)$.

If $(H, \diamond)$ is a simple head type then we usually just write $\diamond$ instead of $(H, \diamond)$.

Each flat bottomed extension of a Λ-algebra and each strictly regular bottomed Λ-algebra is associated with a simple head type: Let $H_b = \mathbb{T}$ for each $b \in B$ and for each $k \in K$ let $\diamond_k^\bot : H^{k^\triangleright} \to H_{k_\triangleleft}$ be the mapping defined by

$$\diamond_k^\bot(v) = \begin{cases} \natural & \text{if } v = \natural^k, \\ \bot & \text{otherwise,} \end{cases}$$

where $\natural^k$ is the element of the set $H^{k^\triangleright}$ defined by $\natural^k(\eta) = \natural$ for each $\eta \in \langle k^\triangleright \rangle$, and let $\diamond_k^\natural : H^{k^\triangleright} \to H_{k_\triangleleft}$ be given by $\diamond_k^\natural(v) = \natural$ for all $v \in H^{k^\triangleright}$. Then $\diamond^\bot$ and $\diamond^\natural$ are both $\natural$-invariant head types and the following holds:

Proposition 3.3.3 *(1) A bottomed Λ-algebra is a $\diamond^\bot$-algebra if and only if it is the flat bottomed extension of some Λ-algebra.*

(2) A bottomed Λ-algebra (X, p) is a $\diamond^\natural$-algebra if and only if $p_k(v) \neq \bot_{k_\triangleleft}$ for all $v \in X^{k^\triangleright}$, $k \in K$. In particular, each strictly regular bottomed Λ-algebra is a $\diamond^\natural$-algebra.

Proof (1) If (X, p) is a $\diamond^\bot$-algebra then by Proposition 3.3.2 $X^\natural$ is invariant in (X, p), and it is easy to check that (X, p) is then the flat bottomed extension of the subalgebra associated with $X^\natural$. Conversely, the flat bottomed extension of a Λ-algebra is clearly a $\diamond^\bot$-algebra.

(2) This is clear. $\square$

Example 3.3.1 The bottomed Λ-algebra (Y, q) given in Example 3.1.2 is a V-minimal regular $\diamond^\bot$-algebra.

The bottomed Λ-algebra (Y, q) in Example 3.1.3 is a V-minimal regular $\diamond^\natural$-algebra.

Besides $\diamond^\bot$ and $\diamond^\natural$ there are two further simple head types which should perhaps be mentioned: One is the 'degenerate' head type $\diamond^\flat$, where $\diamond_k^\flat$ is the constant

mapping with value $\perp$ for each $k \in K$ (and so in particular $\diamond^\flat$ is not $\natural$-invariant). A bottomed Λ-algebra (X,p) is clearly a $\diamond^\flat$-algebra if and only if $p_k(v) = \perp_{k_\triangleleft}$ for all $v \in X^{k^\triangleright}$, $k \in K$. Of course, such algebras are of little practical use.

Now for the fourth head type: A type $b \in B$ is called a *product type* if K_b consists of exactly one constructor name k with $\langle k^\triangleright \rangle$ containing at least two elements and $k^\triangleright\eta \neq b$ for each $\eta \in \langle k^\triangleright \rangle$. (In the signature Λ in Example 2.2.3, for instance, `pair` is a product type.) Define a simple head type $\diamond^\bowtie$ by letting

$$\diamond_k^\bowtie = \begin{cases} \diamond_k^o & \text{if } k \text{ is the single constructor for some product type,} \\ \diamond_k^\natural & \text{otherwise,} \end{cases}$$

where if $k \in K$ with $k^\triangleright \neq \varepsilon$ then $\diamond_k^o : H^{k^\triangleright} \to H_{k_\triangleleft}$ is given by

$$\diamond_k^o(v) = \begin{cases} \natural & \text{if } v(\eta) = \natural \text{ for at least one } \eta \in \langle k^\triangleright \rangle, \\ \perp & \text{otherwise.} \end{cases}$$

Then $\diamond^\bowtie$ is a $\natural$-invariant head type which is sometimes used instead of $\diamond^\natural$. The reason for perhaps preferring $\diamond^\bowtie$ to $\diamond^\natural$ is that it provides more 'natural' bottomed products. This can be seen by looking at Example 3.3.2 and comparing it with Example 3.1.2.

It is worth noting the following two explicit cases of Proposition 3.3.1. Let V be an A-family of bottomed sets.

Proposition 3.3.4 *The flat bottomed extension of any $V^\natural$-initial Λ-algebra is a V-minimal regular $\diamond^\perp$-algebra. Furthermore, each V-initial bottomed Λ-algebra is a V-minimal regular $\diamond^\natural$-algebra.*

Proof This is left for the reader. □

Example 3.3.2 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 2.2.3 (so $A = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$) and let V be an A-family of bottomed sets. Define a bottomed Λ-algebra (Y, q) bound to V exactly as in Example 3.1.3 except that $Y_{\mathtt{pair}}$ is now defined to be

$$V_{\mathtt{x}} \times V_{\mathtt{y}} \;\; \text{with } \perp_{\mathtt{pair}} = (\perp_{\mathtt{x}}, \perp_{\mathtt{y}})$$

(instead of being $V_{\mathtt{x}} \times V_{\mathtt{y}} \cup \{\perp_{\mathtt{pair}}\}$). The reader is left to show that (Y, q) is a V-minimal regular $\diamond^\bowtie$-algebra.

There is a condition, being $\natural$-stable, which plays an important role in Chapter 5 and which is strongly related to a certain class of simple head types. A bottomed

Λ-algebra (Y, q) is called $\natural$-*stable* if whenever $k \in K$ and v_1, $v_2 \in Y^{k^\triangleright}$ are such that $q_k(v_1) \in Y^\natural_{k_\triangleleft}$ and $v_2(\eta) \in Y^\natural_{k^\triangleright\eta}$ for all $\eta \in \langle k^\triangleright \rangle$ with $v_1(\eta) \in Y^\natural_{k^\triangleright\eta}$ then also $q_k(v_2) \in Y^\natural_{k_\triangleleft}$. Expressed in terms of the contraposition, this says that if $q_k(v_2) = \bot_{k_\triangleleft}$ and v_1 is at least as bad as v_2 in the sense that $v_1(\eta) = \bot_{k^\triangleright\eta}$ whenever $v_2(\eta) = \bot_{k^\triangleright\eta}$ then it must also be the case that $q_k(v_1) = \bot_{k_\triangleleft}$. A head type is said to be $\natural$-*stable* if it is a $\natural$-stable bottomed Λ-algebra, and in particular the simple head types $\diamond^\perp$, $\diamond^\natural$ and $\diamond^\bowtie$ are all $\natural$-stable (as well as the degenerate head type $\diamond^\flat$). The following result shows that $\natural$-stable algebras are not as intractable as it might first appear.

Proposition 3.3.5 *If $(H, \diamond)$ is a $\natural$-stable head type then any $(H, \diamond)$-algebra is $\natural$-stable. Conversely, each $\natural$-stable bottomed Λ-algebra is a $\diamond$-algebra for some $\natural$-stable simple head type $\diamond$.*

Proof Let $(H, \diamond)$ be a $\natural$-stable head type and (Y, q) be a $(H, \diamond)$-algebra; there thus exists a proper bottomed homomorphism $\pi : (Y, q) \to (H, \diamond)$ Let v_1, $v_2 \in Y^{k^\triangleright}$ with $q_k(v_1) \in Y^\natural_{k_\triangleleft}$ and such that $v_2(\eta) \in Y^\natural_{k^\triangleright\eta}$ for all $\eta \in \langle k^\triangleright \rangle$ with $v_1(\eta) = Y^\natural_{k^\triangleright\eta}$. Put $w_1 = \pi^{k^\triangleright}(v_1)$, $w_2 = \pi^{k^\triangleright}(v_2)$; then w_1, $w_2 \in H^{k^\triangleright}$ with $\diamond_k(w_i) = \pi_{k_\triangleleft}(q_k(v_i))$ and $w_i(\eta) = \pi_{k^\triangleright\eta}(v_i(\eta))$ for all $\eta \in \langle k^\triangleright \rangle$, $i = 1, 2$. Therefore $\diamond_k(w_1) \in H^\natural_{k_\triangleleft}$ and $w_2(\eta) \in H^\natural_{k^\triangleright\eta}$ for all $\eta \in \langle k^\triangleright \rangle$ with $w_1(\eta) \in H^\natural_{k^\triangleright\eta}$ and, since $(H, \diamond)$ is $\natural$-stable, this implies that $\pi_{k_\triangleleft}(q_k(v_2)) = \diamond_k(w_2) \in H^\natural_{k_\triangleleft}$. Hence $q_k(v_2) \in Y^\natural_{k_\triangleleft}$ and thus (Y, q) is $\natural$-stable.

Conversely, suppose (Y, q) is $\natural$-stable, for each $b \in B$ let $H_b = \mathbb{T}$ and $\varepsilon_b : Y_b \to H_b$ be the mapping with $\varepsilon_b(\bot_b) = \bot$ and $\varepsilon_b(y) = \natural$ for each $y \in Y^\natural_b$. Let $k \in K$; if v_1, $v_2 \in Y^{k^\triangleright}$ with $\varepsilon^{k^\triangleright}(v_1) = \varepsilon^{k^\triangleright}(v_2)$ then, since (Y, q) is $\natural$-stable, it follows that $\varepsilon_{k_\triangleleft}(q_k(v_1)) = \varepsilon_{k_\triangleleft}(q_k(v_2))$. There thus exists a mapping $\diamond_k : H^{k^\triangleright} \to H_{k_\triangleleft}$ such that $\diamond_k(\varepsilon^{k^\triangleright}(v)) = \varepsilon_{k_\triangleleft}(q_k(v))$ for each $v \in Y^{k^\triangleright}$. Moreover, under the additional condition that $\diamond_k(w) = \natural$ for each $w \in H^{k^\triangleright} \setminus \Im(\varepsilon^{k^\triangleright})$, this mapping $\diamond_k$ is unique. (Note that the additional condition is needed if the mapping $\varepsilon^{k^\triangleright}$ is not surjective, which is possible if $Y_b = \{\bot_b\}$ for some $b \in B$.) Then $\diamond$ is a simple head type and by construction (Y, q) is a $\diamond$-algebra. Furthermore $(H, \diamond)$ is $\natural$-stable: This is more-or-less clear if the mapping ε_b is surjective for each $b \in B$. The general case is left for the reader.

Proposition 3.3.6 *Let $(H, \diamond)$ be a $\natural$-stable head type and V be an A-family of bottomed sets. Then any V-minimal regular $(H, \diamond)$-algebra (X, p) is intrinsically free: For each $(H, \diamond)$-algebra (Y, q) and each family $\tau : V \to Y_{|A}$ of bottomed mappings there exists a unique bottomed homomorphism $\pi : (X, p) \to (Y, q)$ with $\pi_{|A} = \tau$.*

Proof As in the proof of Proposition 3.2.2 let (Z, r) be a V-initial bottomed Λ-algebra, let λ be the unique bottomed homomorphism from (Z, r) to (X, p)

fixing V and let $\mu : (Z, r) \to (Y, q)$ be the unique bottomed homomorphism such that $\mu_{|A} = \tau$. Moreover, again let M, $N \subset Z$ be the families defined by $M_b = \{z \in Z_b : \lambda_b(z) = \bot_b\}$ and $N_b = \{z \in Z_b : \mu_b(z) = \bot_b\}$ for each $b \in B$. Then by Lemma 3.2.7 it is enough to show that $M \subset N$. Moreover, the proof of Lemma 3.2.8 showed that $M = L$, where $L_b = \{z \in Z_b : \delta_b(z) = \bot_b\}$ for each $b \in B$ and where $\delta : (Z, r) \to (H, \diamond)$ is the unique bottomed homomorphism such that $\delta_a(z) = \natural$ for all $z \in V_a^\natural$, $a \in A$,

Now by assumption there exists a proper bottomed homomorphism α from (Y, q) to $(H, \diamond)$. Thus $\gamma = \alpha \circ \mu : (Z, r) \to (H, \diamond)$ is a bottomed homomorphism and $N = L'$, where $L'_b = \{z \in Z_b : \gamma_b(z) = \bot_b\}$ for each $b \in B$ (since the mappings in the family α are proper). For each $b \in B$ let

$$D_b = \{z \in Z_b : \gamma_b(z) = \bot_b \text{ whenever } \delta_b(z) = \bot_b\} \;;$$

then the family D is bottomed and $D_{|A} = V$, since $L_a = \{\bot_a\}$ for each $a \in A$. Moreover, D is invariant in (Z, r): Let $k \in K$ and $v \in D^{k^\triangleright}$; for each $\eta \in \langle k^\triangleright \rangle$ we then have $\gamma^{k^\triangleright}(v)(\eta) = \gamma_{k^\triangleright\eta}(v(\eta)) = \bot_{k^\triangleright\eta}$ whenever $\delta^{k^\triangleright}(v)(\eta) = \delta_{k^\triangleright\eta}(v(\eta)) = \bot_{k^\triangleright\eta}$ and thus $\gamma_{k_\triangleleft}(r_k(v)) = \diamond_k(\gamma^{k^\triangleright}(v)) = \bot_{k_\triangleleft}$ whenever $\delta_{k_\triangleleft}(r_k(v)) = \diamond_k(\delta^{k^\triangleright}(v)) = \bot_{k_\triangleleft}$, since $(H, \diamond)$ is $\natural$-stable, i.e., $r_k(v) \in D_{k_\triangleleft}$. Therefore $D = X$, since (X, p) is V-minimal (by Proposition 3.1.3), and this implies that $M = L \subset L' = N$. $\square$

As usual we end the section by considering the case when Λ is the disjoint union of the signatures Λ_i, $i \in F$. Let $(H, \diamond)$ be a head type for the signature Λ. Then by Lemma 2.2.1 $(H, \diamond) = \oplus_{i \in F}(H^i, \diamond^i)$, where $H^i = H_{|B_i}$ and $\diamond^i = \diamond_{|K_i}$ for each $i \in F$, and clearly $(H^i, \diamond^i)$ is a head type for the signature Λ_i.

Proposition 3.3.7 *For each $i \in F$ let (X^i, p^i) be an $(H^i, \diamond^i)$-algebra. Then the sum $\oplus_{i \in F}(X^i, p^i)$ is an $(H, \diamond)$-algebra.*

Proof This follows immediately from Lemma 2.2.2. $\square$

Note that if $(H, \diamond)$ is $\natural$-invariant (resp. $\natural$-stable) then so is $(H^i, \diamond^i)$ for each $i \in F$. Moreover, the same holds for the property of being a simple head type.

Chapter 4

Partially ordered sets

In this chapter we present those basic facts about partially ordered sets which will be needed in the subsequent chapters. Section 4.1 gives some elementary results about bottomed partially ordered sets (posets). Then in Section 4.2 the corresponding results for complete posets are presented. Section 4.3 discusses the initial completion of posets and their relation to what are called algebraic posets. The presentation in Section 4.3 more-or-less follows that in Wright, Wagner and Thatcher [7]. Initial completions are really just ideal completions, a concept that goes back to Birkhoff [2] (which was first published in 1940).

The use of complete posets as a tool for dealing with the denotational semantics of programming languages was originated by Dana Scott in the late sixties. The reader interested in finding out more about this topic should consult Scott and Gunter [16]; an account of its origins can be found in Scott [15].

4.1 Bottomed partially ordered sets

A *partial order* $\sqsubseteq$ on a set X is a binary relation satisfying:

(1) $x \sqsubseteq x$ for all $x \in X$.

(2) If $x_1 \sqsubseteq x_2$ and $x_2 \sqsubseteq x_1$ then $x_1 = x_2$.

(3) If $x_1 \sqsubseteq x_2$ and $x_2 \sqsubseteq x_3$ then $x_1 \sqsubseteq x_3$.

If $\sqsubseteq$ is a partial order on a set X then there can be at most one element $\perp_X \in X$ with $\perp_X \sqsubseteq x$ for all $x \in X$. If such an element $\perp_X$ exists then $\sqsubseteq$ is said to be a *bottomed partial order* and $(X, \sqsubseteq)$ is said to be a *bottomed partially ordered set* with *bottom element* $\perp_X$. Bottomed partially ordered sets will always be referred to simply as *posets* in this study. The reader should be warned, however, that

the existence of a bottom element is not usually taken to be part of the definition of a poset.

We mostly just write X instead of $(X, \sqsubseteq)$ and assume $\sqsubseteq$ can be determined from the context. Something like 'X is a poset with partial order $\sqsubseteq$' or '$\sqsubseteq$ is the partial order on X' can be employed when it is necessary to refer to the partial order explicitly. It is useful to consider the set $\mathbb{I}$ as a poset (naturally with respect to the unique partial order on $\mathbb{I}$).

If X_1 and X_2 are posets then a mapping $h : X_1 \to X_2$ is said to be *monotone* if $h(x) \sqsubseteq_2 h(x')$ for all $x, x' \in X_1$ with $x \sqsubseteq_1 x'$ (where $\sqsubseteq_j$ is the partial order on X_j for $j = 1, 2$). A monotone mapping h to be *bottomed* (or *strict*) if $h(\perp_1) = \perp_2$.

Proposition 4.1.1 *(1) For each poset X the identity mapping* $\mathrm{id}_X : X \to X$ *is monotone.*

(2) If X_1, X_2, X_3 are posets and $g : X_1 \to X_2$ and $h : X_2 \to X_3$ are monotone mappings then the mapping $h \circ g : X_1 \to X_3$ is also monotone.

Proof This is clear. □

Proposition 4.1.1 implies there is a category whose objects are posets and whose morphisms are monotone mappings. This category will be denoted (along with its objects) by Posets. Note that the morphisms in Posets are not assumed to be bottomed.

A monotone mapping $h : X_1 \to X_2$ is said to be an *order isomorphism* if it is an isomorphism in the category Posets, i.e., if there exists a monotone mapping $g : X_2 \to X_1$ such that $g \circ h = \mathrm{id}_{X_1}$ and $h \circ g = \mathrm{id}_{X_2}$. It is clear that an order isomorphism is bottomed. An order isomorphism is of course a bijective mapping, but the set-theoretic inverse of a bijective monotone mapping $h : X_1 \to X_2$ need not be monotone (even if X_1 and X_2 are sets with only three elements).

If X is a poset with bottom element $\perp_X$ then the bottomed set $(X, \perp_X)$ will be referred to as the *underlying bottomed set* of X. This bottomed set will usually also be denoted just by X, the particular usage of the symbol X being determined by the context. If X is a bottomed set with bottom element $\perp_X$ then by a *partial order* $\sqsubseteq$ on X is meant a partial order on the set X with $\perp_X \sqsubseteq x$ for all $x \in X$.

A poset X is said to be *flat* if $x_1 \sqsubseteq x_2$ if and only $x_1 \in \{\perp_X, x_2\}$. Such a poset is of course determined uniquely by its underlying bottom set, and if X is a bottomed set then the partial order defined in this way will be referred to as the *flat* partial order on X.

If X is an S-family of posets then, unless something explicit to the contrary is stated, the partial order on X_s will always be denoted by $\sqsubseteq_s$ and the bottom element by $\perp_s$ for each $s \in S$. If X is a finite S-family of posets then there is

a partial order $\sqsubseteq$ on the set $\otimes X$ defined by stipulating that $v \sqsubseteq v'$ if and only if $v(s) \sqsubseteq_s v'(s)$ for each $s \in S$. The element $\perp \in \otimes X$ with $\perp(s) = \perp_s$ for all $s \in S$ then satisfies $\perp \sqsubseteq v$ for all $v \in \otimes X$, and so $(\otimes X, \sqsubseteq)$ is a poset with bottom element $\perp$. A reference to the poset $\otimes X$ always means with this partial order. Note that if $\grave{X}$ is the corresponding S-family of bottomed sets (i.e., $\grave{X}_s$ is the underlying bottomed set of X_s for each $s \in S$) then $\otimes \grave{X}$ is the underlying bottomed set of $\otimes X$.

Let $n \geq 2$ and for each $j = 1, \ldots, n$ let $(X_j, \sqsubseteq_j)$ be a poset. Then the above construction produces the usual *product partial order* $\sqsubseteq$ on $X_1 \times \cdots \times X_n$ in which $(x_1, \ldots, x_n) \sqsubseteq (x'_1, \ldots, x'_n)$ if and only if $x_j \sqsubseteq_j x'_j$ for each j.

As with the categories Sets and BSets the process of taking finite products results in a mapping $\otimes : \mathcal{T}(\mathsf{Posets}) \to \mathsf{Posets}$. If S is an arbitrary set, X an S-family of posets and γ a finite S-typing then the poset $\otimes(X \circ \gamma)$ will be denoted by X^γ.

Again let X be a finite S-family of posets.

Lemma 4.1.1 *For each $s \in S$ the projection mapping $\mathrm{p}_s : \otimes X \to X_s$ defined by $\mathrm{p}_s(v) = v(s)$ for each $v \in \otimes X$ is monotone.*

Proof This is clear. □

Proposition 4.1.2 *A mapping $h : Y \to \otimes X$ from a poset Y to the poset $\otimes X$ is monotone if and only if for each $s \in S$ the mapping $\mathrm{p}_s \circ h$ from Y to X_s is monotone.*

Proof Straightforward. □

Proposition 4.1.3 *Let Y be a further S-family of posets and let $\varphi : X \to Y$ be an S-family of monotone mappings. Then the mapping $\otimes\varphi : \otimes X \to \otimes Y$ is monotone.*

Proof If v, $v' \in \otimes X$ with $v \sqsubseteq v'$ then for each $s \in S$

$$\otimes\varphi(v)(s) = \varphi_s(v(s)) \sqsubseteq_s \varphi_s(v'(s)) = \otimes\varphi(v')(s) \ ,$$

since φ_s is monotone, and hence $\otimes\varphi$ is monotone. □

The following is a special case of Proposition 4.1.3: Let $n \geq 2$ and for $j = 1, \ldots, n$ let X_j and X'_j be posets and $h_j : X_j \to X'_j$ be a monotone mapping. Then the mapping $h : X_1 \times \cdots \times X_n \to X'_1 \times \cdots \times X'_n$ defined by

$$h(x_1, \ldots, x_n) = (h_1(x_1), \ldots, h_n(x_n))$$

for each $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ is monotone.

4.2 Complete posets

We now define what it means for a poset to be complete and what it means for a monotone mapping between complete posets to be continuous. We then show that the results presented in Section 4.1 hold for complete posets with 'monotone' replaced by 'continuous'. Recall that 'poset' means a partially ordered set with a bottom element.

Let X be a poset with partial order $\sqsubseteq$ and let D be a non-empty subset of X. An element $x \in X$ is called an *upper bound* of D if $x' \sqsubseteq x$ for all $x' \in D$; x is called the *least upper bound* of D if x is an upper bound of D and $x \sqsubseteq x'$ for each upper bound x' of D. (It is clear that there can be at most one element $x \in X$ having these properties.) If the least upper bound of D exists then it will be denoted by $\bigsqcup D$.

A subset D of a poset X is said to be *directed* if it is non-empty and for each $x_1, x_2 \in D$ there exists $x \in D$ such that $x_1 \sqsubseteq x$ and $x_2 \sqsubseteq x$. The set of directed subsets of X will be denoted by $\mathrm{d}(X)$. If $h : X_1 \to X_2$ is monotone then clearly $h(D) \in \mathrm{d}(X_2)$ for each $D \in \mathrm{d}(X_1)$.

A poset X is said to be *complete* if the least upper bound $\bigsqcup D$ of D exists for each $D \in \mathrm{d}(X)$. If X_1 and X_2 are complete posets then a mapping $h : X_1 \to X_2$ is said to be *continuous* if h is monotone and $h(\bigsqcup D) = \bigsqcup h(D)$ for each $D \in \mathrm{d}(X_1)$. Note that if $h : X_1 \to X_2$ is monotone then $\bigsqcup h(D) \sqsubseteq_2 h(\bigsqcup D)$ always holds (with $\sqsubseteq_2$ the partial order on X_2). Thus a monotone mapping h is continuous if and only if $h(\bigsqcup D) \sqsubseteq_2 \bigsqcup h(D)$ for each $D \in \mathrm{d}(X_1)$.

Proposition 4.2.1 *(1) The identity mapping* $\mathrm{id}_X : X \to X$ *is continuous for each complete poset* X.

(2) If X_1, X_2 *and* X_3 *are complete posets and* $g : X_1 \to X_2$ *and* $h : X_2 \to X_3$ *are continuous mappings then the mapping* $h \circ g : X_1 \to X_3$ *is also continuous.*

Proof This is clear. □

Proposition 4.2.1 implies there is a category whose objects are complete posets and whose morphisms are continuous mappings. This category will be denoted (along with its objects) by $\mathsf{CPosets}$. Of course, $\mathsf{CPosets}$ is in fact a subcategory of Posets.

Lemma 4.2.1 *Let* X *be a complete poset and* Y *a poset isomorphic to* X*; then* Y *is also complete. Moreover, if* $h : Y \to X$ *is any order isomorphism then* h *and the inverse mapping* $h^{-1} : X \to Y$ *are both automatically continuous.*

Proof Let $D \in \mathrm{d}(Y)$; then $D' = h(D)$ is an element of $\mathrm{d}(X)$, thus let $x = \bigsqcup D'$ and put $y = h^{-1}(x)$. Now x' is an upper bound of D' if and only if $h^{-1}(x')$ is an upper bound of D, and hence $y = \bigsqcup D$. This shows that Y is complete and that h is continuous (since $h(\bigsqcup D) = h(y) = x = \bigsqcup D' = \bigsqcup h(D)$). The continuity of h^{-1} follows by reversing the roles of X and Y. $\square$

In what follows let X be a finite S-family of complete posets.

Lemma 4.2.2 *Let $D \in \mathrm{d}(\otimes X)$ and for each $s \in S$ put $D_s = \{v(s) : v \in D\}$. Then $D_s \in \mathrm{d}(X_s)$, and the least upper bound of D is the assignment $v \in \otimes X$ defined by $v(s) = \bigsqcup D_s$ for each $s \in S$.*

Proof Straightforward. $\square$

Proposition 4.2.2 *The poset $\otimes X$ is complete.*

Proof This follows immediately from Lemma 4.2.2. $\square$

There is thus a mapping $\otimes : \mathcal{T}(\mathsf{CPosets}) \to \mathsf{CPosets}$ obtained by taking finite products in $\mathsf{CPosets}$. If S is an arbitrary set, X an S-family of complete posets and γ a finite S-typing then the complete poset $\otimes(X \circ \gamma)$ will be denoted by X^γ.

Lemma 4.2.3 *For each $s \in S$ the projection mapping $\mathrm{p}_s : \otimes X \to X_s$ defined by $\mathrm{p}_s(v) = v(s)$ for each $v \in \otimes X$ is continuous.*

Proof This follows immediately from Lemma 4.2.2. $\square$

Proposition 4.2.3 *A mapping $h : Y \to \otimes X$ from a complete poset Y to the complete poset $\otimes X$ is continuous if and only if for each $s \in S$ the mapping $\mathrm{p}_s \circ h$ from Y to X_s is continuous.*

Proof If h is continuous then $\mathrm{p}_s \circ h$ is also continuous for each $s \in S$ (since it is the composition of two continuous mappings). Conversely, suppose that $\mathrm{p}_s \circ h$ is continuous for each $s \in S$; in particular, (as in Proposition 4.1.2 h is then monotone. Let $D \in \mathrm{d}(Y)$; then by Lemma 4.2.2 it follows that for each $s \in S$

$$\begin{aligned}\left(\bigsqcup h(D)\right)(s) &= \bigsqcup \{v(s) : v \in h(D)\} \\ &= \bigsqcup \mathrm{p}_s(h(D)) = (\mathrm{p}_s \circ h)\left(\bigsqcup D\right) = h\left(\bigsqcup D\right)(s)\end{aligned}$$

and so $\bigsqcup h(D) = h(\bigsqcup D)$. This shows h is continuous. $\square$

Proposition 4.2.4 *Let $\varphi : X \to Y$ be a family of continuous mappings (with Y a further S-family of complete posets). Then the mapping $\otimes\varphi : \otimes X \to \otimes Y$ is also continuous.*

Proof By Proposition 4.1.3 the mapping $\otimes\varphi$ is monotone. Let $D \in \mathrm{d}(\otimes X)$; then, with two applications of Lemma 4.2.2 and since φ_s is continuous, it follows that

$$\begin{aligned}\otimes\varphi\left(\bigsqcup D\right)(s) &= \varphi_s\left(\left(\bigsqcup D\right)(s)\right) = \varphi_s\left(\bigsqcup\{v(s) : v \in D\}\right)\\ &= \bigsqcup\{\varphi_\eta(v(s)) : v \in D\} = \bigsqcup\{\otimes\varphi(v)(s) : v \in D\} = \left(\bigsqcup \otimes\varphi(D)\right)(s)\end{aligned}$$

for each $s \in S$, and therefore $\otimes\varphi(\bigsqcup D) = \bigsqcup \otimes\varphi(D)$. Thus $\otimes\varphi$ is continuous. □

Note the following special case of Proposition 4.2.4: Let $n \geq 2$ and for each $j = 1, \ldots, n$ let X_j and X'_j be complete posets and $h_j : X_j \to X'_j$ be a continuous mapping. Then the mapping $h : X_1 \times \cdots \times X_n \to X'_1 \times \cdots \times X'_n$ defined by

$$h(x_1, \ldots, x_n) = (h_1(x_1), \ldots, h_n(x_n))$$

for each $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ is continuous.

4.3 Initial completions and algebraic posets

A poset Y_1 is said to be a *subposet* of a poset Y_2 if $Y_1 \subset Y_2$, Y_1 and Y_2 have a common bottom element and the partial order on Y_1 is obtained by restricting the partial order on Y_2. (This means that if $\sqsubseteq_j$ is the partial order on Y_j for $j = 1, 2$ and $y, y' \in Y_1$ then $y \sqsubseteq_1 y'$ if and only if $y \sqsubseteq_2 y'$.) The poset Y_2 is then said to be an *extension* of Y_1.

A poset X is said to be a *completion* of a poset Y if X is a complete extension of Y and each element of X is the least upper bound (in X) of some element of $\mathrm{d}(Y)$.

Warning Let X be a complete extension of a poset Y and let $D \in \mathrm{d}(Y)$. Then it is possible that D has a least upper bound in Y which is not equal to the least upper bound of D in X. (Note, however, that if the least upper bound of D in X is an element of Y then in this case it must also be the least upper bound of D in Y.) When speaking just of the least upper bound of D or writing $\bigsqcup D$ then the least upper bound in X is always meant.

If Y is a poset with partial order $\sqsubseteq$ and X is an extension of Y then, unless something explicit to the contrary is stated, the partial order on X will also be denoted by $\sqsubseteq$.

Proposition 4.3.1 *Let Y be a finite S-family of posets and for each $s \in S$ let X_s be a completion of Y_s. Then $\otimes X$ is a completion of $\otimes Y$.*

Proof Denote the partial orders on $\otimes Y$ and $\otimes X$ by $\sqsubseteq$. Let $v \in \otimes X$. Then for each $s \in S$ there exists a subset $D_s \in \mathrm{d}(Y_s)$ with $v(s) = \bigsqcup D_s$ (since X_s is a completion of Y_s). Put

$$D = \{u \in \otimes Y : u(s) \in D_s \text{ for each } s \in S\} ;$$

then $D \in \mathrm{d}(\otimes Y)$: If $v_1, v_2 \in D$ then for each $s \in S$ there exists $y_s \in D_s$ with $v_1(s) \sqsubseteq_s y_s$ and $v_2(s) \sqsubseteq_s y_s$; thus if $v \in \otimes Y$ is defined by letting $v(s) = y_s$ for each $s \in S$ then $v \in D$ and $v_1 \sqsubseteq v$, $v_2 \sqsubseteq v$. Moreover, $v = \bigsqcup D$: If $u \in D$ then $u(s) \in D_s$ and so $u(s) \sqsubseteq_s \bigsqcup D_s = v(s)$ for each $s \in S$, i.e., $u \sqsubseteq v$. On the other hand, if $v' \in \otimes X$ is an upper bound of D then $u(s) \sqsubseteq_s v'(s)$ for each $u \in D$ and this implies that $y \sqsubseteq_s v'(s)$ for all $y \in D_s$ (since for each $y \in D_s$ there exists $u \in D$ with $u(s) = y$) and thus that $v(s) = \bigsqcup D_s \sqsubseteq_s v'(s)$ for each $s \in S$, i.e., $v \sqsubseteq v'$. This, together with Proposition 4.2.3, shows that $\otimes X$ is a completion of $\otimes Y$. □

Lemma 4.3.1 *Let X be a completion of a poset Y and let $h : Y \to X'$ be a monotone mapping from Y to a complete poset X'. Then there exists at most one extension of h to a continuous mapping $h' : X \to X'$ (i.e., there exists at most one continuous mapping $h' : X \to X'$ with $h'(y) = h(y)$ for all $y \in Y$).*

Proof Let $x \in X$; then there exists $D \in \mathrm{d}(Y)$ with $x = \bigsqcup D$ and it therefore follows that $h'(x) = \bigsqcup h(Y)$. □

Lemma 4.3.2 *Let X_1 and X_2 be completions of a poset Y. Then there is at most one continuous mapping $h : X_1 \to X_2$ such that $h(y) = y$ for all $y \in Y$. Moreover, if such a mapping exists then it surjective.*

Proof The first statement follows immediately from Lemma 4.3.1. Thus suppose that h exists and let $x_2 \in X_2$; then there exists $D \in \mathrm{d}(Y)$ with $x_2 = \bigsqcup D$ (the least upper bound of D in X_2). Let $x_1 = \bigsqcup D$ (the least upper bound of D in X_1); then $h(x_1) = h(\bigsqcup D) = \bigsqcup h(D) = \bigsqcup D = x_2$, since h is continuous. Hence h is surjective. □

Let Y be a poset, which is considered to be fixed in what follows. Then there is a category whose objects are completions of Y and where if X_1 and X_2 are such completions then the morphisms in $\mathrm{Hom}(X_1, X_2)$ are the continuous mappings $h : X_1 \to X_2$ with $h(y) = y$ for each $y \in Y$. Of course, Lemma 4.3.2 implies that

$\mathrm{Hom}(X_1, X_2)$ can contain at most one morphism for each X_1, X_2. A completion X of Y is said to be *initial* if it is an initial object in this category. Thus by the previous remark, a completion X is initial if for each completion X' of Y there exists a continuous mapping $h : X \to X'$ with $h(y) = y$ for all $y \in Y$.

Proposition 4.3.2 below states that Y possesses an initial completion. For reasons to be explained at the end of the section, this completion is often also referred to as the *ideal completion* of Y.

A seemingly stronger requirement than being initial on a completion of Y is the following: A completion X is said to have the *continuous extension property* if whenever $h : Y \to X'$ is a monotone mapping from Y to a complete poset X' then h can be uniquely extended to a continuous mapping $h' : X \to X'$ (i.e., there exists a unique continuous mapping $h' : X \to X'$ with $h'(y) = h(y)$ for all $y \in Y$). It is clear that a completion X of Y with the continuous extension property is initial, since if X' is any completion of Y then the mapping $i : Y \to X'$ with $i(y) = y$ for all $y \in Y$ is monotone. It turns out that the converse is also true.

Let D_1, $D_2 \in \mathrm{d}(Y)$; then D_1 is said to be *cofinal* in D_2 if for each $y_1 \in D_1$ there exists $y_2 \in D_2$ with $y_1 \sqsubseteq y_2$. If X is a complete extension of Y and D_1, $D_2 \in \mathrm{d}(Y)$ with D_1 cofinal in D_2 then clearly $\bigsqcup D_1 \sqsubseteq \bigsqcup D_2$.

Proposition 4.3.2 *There exists a initial completion of Y, and the following five statements are equivalent for a completion X of Y:*

(1) X is initial.

(2) X has the continuous extension property.

(3) D_1 is cofinal in D_2 whenever D_1, $D_2 \in \mathrm{d}(Y)$ with $\bigsqcup D_1 \sqsubseteq \bigsqcup D_2$.

(4) $y \sqsubseteq y'$ for some $y' \in D$ whenever $y \in Y$ and $D \in \mathrm{d}(Y)$ with $y \sqsubseteq \bigsqcup D$.

(5) $y \sqsubseteq x$ for some $x \in D$ whenever $y \in Y$ and $D \in \mathrm{d}(X)$ with $y \sqsubseteq \bigsqcup D$.

Proof Let $\sqsubseteq$ be the partial order on Y. It will be shown first that there exists a completion X of Y for which statement (3) holds, i.e., such that D_1 is cofinal in D_2 whenever D_1, $D_2 \in \mathrm{d}(Y)$ with $\bigsqcup D_1 \sqsubseteq \bigsqcup D_2$.

Elements D_1, $D_2 \in \mathrm{d}(Y)$ are said to be *mutually cofinal* if each is cofinal in the other. Mutual cofinality clearly defines an equivalence relation on the set $\mathrm{d}(Y)$. Let X denote the set of equivalence classes, and if $D \in \mathrm{d}(Y)$ then denote by $[D]$ the equivalence class containing D. If D_1 is equivalent to D_1' and D_2 equivalent to D_2' then clearly D_1 is cofinal in D_2 if and only if D_1' is cofinal in D_2'. Thus define a relation $\preceq$ on X by letting $[D_1] \preceq [D_2]$ if D_1 is cofinal in D_2; it is immediate that $\preceq$ is a partial order.

Now $\{y\} \in \mathrm{d}(Y)$ for each $y \in Y$, which means a mapping $i : Y \to X$ can be defined by putting $i(y) = [\{y\}]$ for each $y \in Y$. Then i is an *embedding*, i.e.,

$y_1 \sqsubseteq y_2$ holds if and only if $i(y_1) \preceq i(y_2)$ (since $y_1 \sqsubseteq y_2$ if and only if $\{y_1\}$ is cofinal in $\{y_2\}$), and Y can thus be considered as a subposet of X by identifying Y with $i(Y)$. But this just means that X is an extension of Y.

Lemma 4.3.3 *The poset X (with the partial order $\preceq$) is a completion of Y such that if $D_1, D_2 \in \mathrm{d}(Y)$ with $\bigsqcup D_1 \sqsubseteq \bigsqcup D_2$ then D_1 is cofinal in D_2.*

Proof Let $C \in \mathrm{d}(X)$. For each $x \in C$ choose $D_x \in \mathrm{d}(Y)$ with $[D_x] = x$ and put $D = \bigcup_{x \in C} D_x$. Then $D \in \mathrm{d}(Y)$. (Let $y_1, y_2 \in D$; then there exist $x_1, x_2 \in C$ such that $y_1 \in D_{x_1}$ and $y_2 \in D_{x_2}$ and, since $C \in \mathrm{d}(X)$, there exists $x \in C$ with $x_1 \preceq x$ and $x_2 \preceq x$. But this means that $y_1', y_2' \in D_x$ can be found with $y_1 \sqsubseteq y_1'$ and $y_2 \sqsubseteq y_2'$ and hence there exists $y \in D_x \subset D$ such that $y_1 \sqsubseteq y_1' \sqsubseteq y$ and $y_2 \sqsubseteq y_2' \sqsubseteq y$, because $D_x \in \mathrm{d}(Y)$.) It is now enough to show that $\hat{x} = [D]$ is the least upper bound of C in X. If $x \in C$ and $y \in D_x$ then $y \sqsubseteq y$ and $y \in D$; thus D_x is cofinal in D. This implies that $x \preceq \hat{x}$, i.e., $\hat{x}$ is an upper bound of C. Now let $x' = [D']$ be any upper bound of C. If $y \in D$ then $y \in D_x$ for some $x \in C$ and D_x is cofinal in D', since $x \preceq x'$. There therefore exists $y' \in D'$ with $y \sqsubseteq y'$ and hence D is cofinal in D', i.e., $\hat{x} \preceq x'$. The poset X is therefore complete.

Now if $x = [D] \in X$ and $C = \bigcup_{y \in D} \{y\}$ then C is a directed subset of $Y \subset X$ and, as above, $x = \bigsqcup C$. This shows X is a completion of Y. Finally, if $D_1, D_2 \in \mathrm{d}(Y)$ with $\bigsqcup D_1 \preceq \bigsqcup D_2$ then $[D_1] \preceq [D_2]$, since $\bigsqcup D = [D]$ for each $D \in \mathrm{d}(Y)$, which by definition means that D_1 is cofinal in D_2. □

It will be shown next that any completion of Y for which statement (3) holds has the continuous extension property. We first establish the following fact:

Lemma 4.3.4 *Let X be a complete extension of Y (although not necessarily a completion of Y) and suppose that whenever D_1, D_2 are elements of $\mathrm{d}(Y)$ with $\bigsqcup D_1 \sqsubseteq \bigsqcup D_2$ then D_1 is cofinal in D_2. Let X' denote the set of all elements of X having the form $\bigsqcup D$ for some $D \in \mathrm{d}(Y)$. Then for each $D' \in \mathrm{d}(X')$ there exists $D \in \mathrm{d}(Y)$ with D cofinal in D' and $\bigsqcup D = \bigsqcup D'$.*

Proof Let $D' \in \mathrm{d}(X')$ and for each $x \in D'$ choose $D_x \in \mathrm{d}(Y)$ with $x = \bigsqcup D_x$. Then $D = \bigcup_{x \in D'} D_x \in \mathrm{d}(Y)$. To see this let $y_1, y_2 \in D$; then there exist $x_1, x_2 \in D'$ with $y_1 \in D_{x_1}$ and $y_2 \in D_{x_2}$ and, since $D' \in \mathrm{d}(X')$, there exists $x \in D'$ such that $x_1 \sqsubseteq x$ and $x_2 \sqsubseteq x$. But then by assumption D_{x_1} is cofinal in D_x and D_{x_2} is cofinal in D_x. Hence $y_1', y_2' \in D_x$ can be found with $y_1 \sqsubseteq y_1'$ and $y_2 \sqsubseteq y_2'$ and, since $D_x \in \mathrm{d}(Y)$, it follows that $y_1 \sqsubseteq y_1' \sqsubseteq y$ and $y_2 \sqsubseteq y_2' \sqsubseteq y$ for some $y \in D_x \subset D$. Now if $y \in D$ then $y \in D_x$ for some $x \in D'$, and so $y \sqsubseteq x$; this implies D is cofinal in D' and thus also that $\bigsqcup D \sqsubseteq \bigsqcup D'$. On the other hand, if $x \in D'$ then $x \sqsubseteq \bigsqcup D$, since $x = \bigsqcup D_x$ and $D_x \subset D$, and therefore $\bigsqcup D' \sqsubseteq \bigsqcup D$. Hence $\bigsqcup D = \bigsqcup D'$. □

Lemma 4.3.5 *Let X be a completion of Y for which statement (3) holds. Then X has the continuous extension property.*

Proof Let $h : Y \to X'$ be a monotone mapping of Y to a complete poset X'. If $D_1, D_2 \in \mathrm{d}(Y)$ with $\bigsqcup D_1 = \bigsqcup D_2$ then each of D_1 and D_2 is cofinal in the other and therefore $\bigsqcup h(D_1) = \bigsqcup h(D_2)$. Thus, since each element of X has the form $\bigsqcup D$ for some $D \in \mathrm{d}(Y)$, a mapping $h' : X \to X'$ can be defined by putting $h'(x) = \bigsqcup h(D)$, where D is any element of $\mathrm{d}(Y)$ with $x = \bigsqcup D$. Then h' is monotone and an extension of h, because $\{y\} \in \mathrm{d}(Y)$ and $\bigsqcup\{y\} = y$ for each $y \in Y$. Now in order to show the mapping h' is continuous it is enough to show that $h'(\bigsqcup D') \sqsubseteq \bigsqcup h'(D')$ for each $D' \in \mathrm{d}(X)$ (with $\sqsubseteq$ also denoting the partial order on X'). Let $D' \in \mathrm{d}(X)$; then by Lemma 4.3.4 there exists $D \in \mathrm{d}(Y)$ with D cofinal in D' and $\bigsqcup D = \bigsqcup D'$. Hence

$$h'\left(\bigsqcup D'\right) = h'\left(\bigsqcup D\right) = \bigsqcup h(D) = \bigsqcup h'(D) \sqsubseteq \bigsqcup h'(D') .$$

Finally, Lemma 4.3.1 implies that this extension h' of h is unique. □

It was already noted that a completion having the continuous extension property is initial. Together with Lemmas 4.3.3 and 4.3.5 this implies that there exists an initial extension of Y.

Lemma 4.3.6 *Statement (3) holds for any initial completion of Y.*

Proof It has already been seen that there exists a completion for which statement (3) holds and that any such completion is initial. Moreover, it is easily checked that statement (3) holds for any completion isomorphic to a completion for which this is the case. Thus by Proposition 2.1.2 statement (3) holds for any initial completion of Y. □

It has now been established that statements (1), (2) and (3) are equivalent. But it is clear that (3) and (4) are equivalent and that (5) implies (4); it thus remains to show that the last statement is implied by the others and this follows immediately from Lemma 4.3.4. This completes the proof of Proposition 4.3.2. □

Warning If X is an initial completion of a poset Y then in general X is not an initial completion of itself. In fact, the class of complete posets which are initial completions of themselves is very special and will be characterised below in Proposition 4.3.4.

Proposition 4.3.3 *Let S be a finite set, let Y be a family of posets, and for each $s \in S$ let X_s be an initial completion of Y_s. Then $\otimes X$ is an initial completion of $\otimes Y$.*

Proof Denote the partial orders on $\otimes Y$ and $\otimes X$ by $\sqsubseteq$. By Proposition 4.3.1 $\otimes X$ is a completion of $\otimes Y$. Let D and D' be elements of $\mathrm{d}(\otimes Y)$ with $\bigsqcup D \sqsubseteq \bigsqcup D'$. For each $s \in S$ let $D_s = \{u(s) : u \in D\}$ and $D'_s = \{u(s) : u \in D'\}$. Then $D_s, D'_s \in \mathrm{d}(Y_s)$ and $\bigsqcup D_s \sqsubseteq_s \bigsqcup D'_s$ and thus, since X_η is initial, D_s is cofinal in D'_s for each $s \in S$. From this it follows that D is cofinal in D': Let $v \in D$; then for each $s \in S$ there exists $v_s \in D'$ with $v(s) \sqsubseteq_s v_s(s)$, and therefore, since S is finite and D' is directed, there exists $v' \in D'$ with $v_s \sqsubseteq v'$ for all $s \in S$. But then $v \sqsubseteq v'$. Hence by Proposition 4.3.2 ((3) $\Rightarrow$ (1)) $\otimes X$ is an initial completion of $\otimes Y$. $\square$

Proposition 4.3.4 *Let X be a complete poset. Then X is an initial completion of itself if and only if each directed subset of X contains a maximum element (i.e., if and only if for each $D \in \mathrm{d}(X)$ then there exists $x \in D$ with $x' \sqsubseteq x$ for all $x' \in D$).*

Proof Suppose first that X is an initial completion of itself and let $D \in \mathrm{d}(X)$ with $\bigsqcup D = x$. Then $x \sqsubseteq \bigsqcup D$ and so by Proposition 4.3.2 ((1) $\Rightarrow$ (5)) $x \sqsubseteq x'$ for some $x' \in D$. But this is only possible if $x = x'$ and therefore $x \in D$, i.e., D contains a maximum element (namely x). The converse follows directly from Proposition 4.3.2 ((5) $\Rightarrow$ (1)). $\square$

If Y is a poset then a sequence of elements $\{y_n\}_{n\geq 0}$ from Y is called a *chain* in Y if $y_n \sqsubseteq y_{n+1}$ for each $n \geq 0$. A chain $\{y_n\}_{n\geq 0}$ is said to be *finite* if $y_m = y_n$ for all $m \geq n$ for some $n \geq 0$. Now let X be a complete poset; if X is an initial completion of itself then by Proposition 4.3.4 each chain in X must be finite (since the elements in a chain form a directed set). The converse is in fact also true: If each chain in X is finite then X is an initial completion of itself. This follows from Proposition 4.3.4 together with a standard application of Zorn's lemma.

We now introduce a condition on posets, being *algebraic*, which characterises those posets which arise as initial completions. Moreover, it turns out that an algebraic poset X is the initial completion of exactly one poset Y, and this poset Y is the set of what are called the *compact* elements of X.

Let X be a complete poset. An element $x \in X$ is said to be *compact* if whenever $D \in \mathrm{d}(X)$ with $x \sqsubseteq \bigsqcup D$ then $x \sqsubseteq x'$ for some $x' \in D$. Note that in particular the bottom element $\perp$ of X is always compact. Let $\mathrm{K}(X)$ denote the set of compact elements of X and for each $x \in X$ put $\mathrm{K}_x = \{y \in \mathrm{K}(X) : y \sqsubseteq x\}$.

A poset X is said to be *algebraic* if it is complete and if for each $x \in X$ the set K_x is directed with $\bigsqcup \mathrm{K}_x = x$.

Proposition 4.3.5 *A poset X is algebraic if and only if it the initial completion of some poset Y. Moreover, in this case $Y = \mathrm{K}(X)$.*

Proof This follows directly from Lemmas 4.3.7, 4.3.8 and 4.3.9. □

Lemma 4.3.7 *If X is an initial completion of a poset Y then $Y = \mathrm{K}(X)$.*

Proof By Proposition 4.3.2 ((1) ⇒ (5)) $Y \subset \mathrm{K}(X)$. Conversely, let $x \in \mathrm{K}(X)$; there thus exists $D \in \mathrm{d}(Y)$ with $x = \bigsqcup D$, since D is a completion of Y. But then $x \sqsubseteq \bigsqcup D$ and so $x \sqsubseteq y$ for some $y \in D$. This is only possible if $x = y$ and hence in particular $x \in Y$, i.e., $\mathrm{K}(X) \subset Y$. □

Lemma 4.3.8 *Let X be an initial completion of a poset Y and for each $x \in X$ let $D_x = \{y \in Y : y \sqsubseteq x\}$. Then $D_x \in \mathrm{d}(Y)$ and $\bigsqcup D_x = x$.*

Proof Let $x \in X$; then, since X is a completion of Y, there exists $D \in \mathrm{d}(Y)$ with $\bigsqcup D = x$. Thus $D \subset D_x$ and so in particular $D_x \neq \varnothing$. Let $y \in D_x$; then $y \sqsubseteq x = \bigsqcup D$ and hence by Proposition 4.3.2 ((1) ⇒ (4)) there exists $y' \in D$ with $y \sqsubseteq y'$. Therefore if $y_1, y_2 \in D_x$ then there exist $y'_1, y'_2 \in D$ with $y_1 \sqsubseteq y'_1$ and $y_2 \sqsubseteq y'_2$. But $D \in \mathrm{d}(Y)$ and so there exists $y' \in D$ with $y'_1 \sqsubseteq y'$ and $y'_1 \sqsubseteq y'$, i.e., there exists $y' \in D_x$ with $y_1 \sqsubseteq y'$ and $y_1 \sqsubseteq y'$. This shows that $D_x \in \mathrm{d}(Y)$. It is now clear that $\bigsqcup D_x = x$, since $x = \bigsqcup D \sqsubseteq \bigsqcup D_x$ and by definition x is an upper bound of D_x. □

Lemmas 4.3.7 and 4.3.8 imply that the initial completion of a poset is algebraic.

Lemma 4.3.9 *Let X be an algebraic poset. Then X is an initial completion of* $\mathrm{K}(X)$.

Proof By definition X is a completion of $\mathrm{K}(X)$ and thus by Proposition 4.3.2 ((5) ⇒ (1)) X is an initial completion of $\mathrm{K}(X)$. □

Proposition 4.3.6 *Let X be a finite S-family of algebraic posets. Then $\otimes X$ is an algebraic poset.*

Proof For each $s \in S$ let $Y_s = \mathrm{K}(X_s)$. By Lemma 4.3.9 X_s is an initial completion of Y_s for each $s \in S$ and thus by Proposition 4.3.3 $\otimes X$ is an initial completion of $\otimes Y$. Therefore by Proposition 4.3.5 $\otimes X$ is algebraic. □

The class of all algebraic posets will be denoted by $\mathsf{APosets}$. Moreover, $\mathsf{APosets}$ will also be used to denote the corresponding full subcategory of $\mathsf{CPosets}$. By Proposition 4.3.6 there is a mapping $\otimes : \mathcal{T}(\mathsf{APosets}) \to \mathsf{APosets}$ obtained by taking finite products in $\mathsf{APosets}$.

The final topic of this section explains why the initial completion also goes under the name of the ideal completion. In what follows let Y be a fixed poset. A non-empty subset I of Y is called an *ideal of* Y if $y \in I$ whenever $y \sqsubseteq y'$ for some $y' \in I$. An ideal I is said to be *directed* if $I \in \mathrm{d}(Y)$. In particular, the set

$$I(y) = \{y' \in Y : y' \sqsubseteq y\}$$

is a directed ideal for each $y \in Y$ (*the principal ideal generated by* y). It is clear that if $y_1, y_2 \in Y$ then $y_1 \sqsubseteq y_2$ if and only if $I(y_1) \subset I(y_2)$. Now denote by $\mathcal{I}_{\mathrm{d}}(Y)$ the set of directed ideals of Y, regarded as a poset with the inclusion ordering. By the above remark $\mathcal{I}_{\mathrm{d}}(Y)$ can be considered as an extension of Y (by identifying the element y with the principal ideal $I(y)$ for each $y \in Y$).

Proposition 4.3.7 *The poset $\mathcal{I}_{\mathrm{d}}(Y)$ is an initial completion (called the ideal completion) of Y.*

Proof It is enough to show that $\mathcal{I}_{\mathrm{d}}(Y)$ is isomorphic to the initial completion constructed in the proof of Proposition 4.3.2. For each non-empty subset D of Y let $I(D) = \{y \in Y : y \sqsubseteq y' \text{ for some } y' \in D\}$.

Lemma 4.3.10 *(1) $I(D)$ is an ideal with $D \subset I(D)$. Moreover, if I is any ideal with $D \subset I$ then $I(D) \subset I$, i.e., $I(D)$ is the smallest ideal containing D.*

(2) If $D \in \mathrm{d}(Y)$ then $I(D) \in \mathrm{d}(Y)$; moreover, D and $I(D)$ are mutually cofinal.

(3) If $D_1, D_2 \in \mathrm{d}(Y)$ then $I(D_1) \subset I(D_2)$ if and only if D_1 is cofinal in D_2.

Proof This is straightforward. □

Now let X be as in the proof of Proposition 4.3.2. Recall that mutual cofinality defines an equivalence relation on the set $\mathrm{d}(Y)$ and that X is the set of equivalence classes, considered as a poset with the partial order $\preceq$ defined by stipulating that $[D_1] \preceq [D_2]$ if D_1 is cofinal in D_2 (and where $[D]$ denotes the equivalence class containing D for each $D \in \mathrm{d}(Y)$). If $D_1, D_2 \in \mathrm{d}(Y)$ then by Lemma 4.3.10 (3) $I(D_1) = I(D_2)$ if and only if D_1 and D_2 are equivalent, thus by Lemma 4.3.10 (2) a mapping $h : X \to \mathcal{I}_{\mathrm{d}}(Y)$ can be defined by letting $h([D]) = I(D)$ for each $D \in \mathrm{d}(Y)$. Moreover, by Lemma 4.3.10 (3) h is an embedding. In fact h is also surjective, since by Lemma 4.3.10 (1) and (2) it follows that $D = I(D) = h([D])$ for each $D \in \mathcal{I}_{\mathrm{d}}(Y)$. Hence h is an order isomorphism. But $h(y) = y$ for each $y \in Y$ (because y is identified with $[\{y\}]$ in X, with $I(y)$ in $\mathcal{I}_{\mathrm{d}}(Y)$, and $I(\{y\}) = I(y)$), and therefore X and $\mathcal{I}_{\mathrm{d}}(Y)$ are isomorphic extensions of Y. □

Chapter 5

Ordered and continuous algebras

This chapter deals with the third and fourth steps in our programme of specifying data objects. The third step is based on the observation that if (X, p) is a regular bottomed Λ-algebra satisfying some additional conditions then for each $b \in B$ there is a unique partial order $\sqsubseteq_b$ on the set X_b such that $x \sqsubseteq_b x'$ can reasonably be interpreted as meaning that x is less-defined than x'. In particular, this should mean that $\perp_b \sqsubseteq x$ for each $x \in X_b$ and the mapping $p_k : X^{k^\triangleright} \to X_{k_\triangleleft}$ be monotone for each $k \in K$.

Considering X_b as a poset with this partial order $\sqsubseteq_b$ results in a B-family of posets X together with a K-family of monotone mappings p, and the pair (X, p) will then be called an ordered Λ-algebra. The fourth step is to complete the posets in the family X, and the appropriate completion here is the initial completion presented in Section 4.3. After the completion has been made and each p_k has been extended to a continuous mapping we end up with a continuous algebra, i.e., a Λ-algebra (Y, q) with a B-family of complete posets Y and a K-family of continuous mappings q. The results in this chapter for $\diamond^\natural$-algebras can be found in Goguen, Thatcher, Wagner and Wright [6] and in Courcelle and Nivat [5].

5.1 Ordered algebras

An *ordered* Λ-*algebra* is any pair (X, p) consisting of a B-family of posets X and a K-family of mappings p such that p_k is a monotone mapping from $X^{k^\triangleright}$ to $X_{k_\triangleleft}$ for each $k \in K$. Recall once again that 'poset' means a partially ordered set with a bottomed element and that $X^{k^\triangleright}$ denotes the poset $\otimes(X \circ k^\triangleright)$. If (X, p) is an ordered Λ-algebra and X_b^o is the bottomed set underlying X_b for each $b \in B$ then (X^o, p) is a bottomed Λ-algebra, called the *underlying bottomed* Λ-*algebra* of (X, p). However, the underlying bottomed set X_b^o will usually just be denoted by X_b, which means that the ordered and the underlying bottomed Λ-algebras are

both denoted by (X, p). What a particular usage of (X, p) refers to will always be clear from the context.

If (X, p) is an ordered Λ-algebra then, unless something to the contrary is stated, the partial order on X_b will always be denoted by $\sqsubseteq_b$ and the bottom element by $\perp_b$. For each $k \in K$ the partial order on $X^{k^\triangleright}$ will be denoted by $\sqsubseteq^{k^\triangleright}$, thus if $v_1, v_2 \in X^{k^\triangleright}$ then $v_1 \sqsubseteq^{k^\triangleright} v_2$ if and only if $v_1(\eta) \sqsubseteq_{k^\triangleright\eta} v_2(\eta)$ for each $\eta \in \langle k^\triangleright \rangle$.

An ordered Λ-algebra (X, p) will be called *intrinsic* if the underlying bottomed Λ-algebra is regular and if whenever $b \in B \setminus A$ and $x_1, x_2 \in X_b^\natural$ then $x_1 \sqsubseteq_b x_2$ if and only if $x_1 = p_k(v_1)$ and $x_2 = p_k(v_2)$ with $k \in K_b$ and $v_2 \sqsubseteq^{k^\triangleright} v_2$. We consider intrinsic ordered algebras as being those for which $x \sqsubseteq_b x'$ can really be interpreted as meaning that x is less-defined than x'.

Example 5.1.1 Let (Y, q) be the initial bottomed Λ-algebra introduced in Example 2.2.1 and for each $b \in B$ define a partial order $\sqsubseteq_b$ on Y_b as follows:

If $a, a' \in Y_{\texttt{bool}}$ then $a \sqsubseteq_{\texttt{bool}} a'$ if and only if $a \in \{\perp_{\texttt{bool}}, a'\}$.

Let $n \in \mathbb{N}$ and $x \in Y_{\texttt{nat}}$. Then $x \sqsubseteq_{\texttt{nat}} n$ if and only if either $x = n$ or $x = m^\perp$ for some $m \leq n$. Moreover, $x \sqsubseteq_{\texttt{nat}} n^\perp$ if and only if $x = m^\perp$ for some $m \leq n$.

If $n, n' \in Y_{\texttt{int}}$ then $n \sqsubseteq_{\texttt{int}} n'$ if and only if $n \in \{\perp_{\texttt{int}}, n'\}$.

If $p, p' \in Y_{\texttt{pair}}$ then $p \sqsubseteq_{\texttt{pair}} p'$ if and only if either $p = \perp_{\texttt{pair}}$ or $p = (x, y)$ and $p' = (x', y')$ with $x \sqsubseteq_{\texttt{int}} x'$ and $y \sqsubseteq_{\texttt{int}} y'$.

If $\ell, \ell' \in Y_{\texttt{list}}$ then $\ell \sqsubseteq_{\texttt{list}} \ell'$ if and only if either
$\ell = z_1 \cdots z_m$, $\ell' = z'_1 \cdots z'_m$ with $z_j \sqsubseteq_{\texttt{int}} z'_j$ for each j, or
$\ell = (z_1 \cdots z_m)^\perp$ and ℓ' either $z'_1 \cdots z'_n$ or $(z'_1 \cdots z'_n)^\perp$, with $m \leq n$ and $z_j \sqsubseteq_{\texttt{int}} z'_j$ for each $j = 1, \ldots, m$.

(Recall here that
$Y_{\texttt{bool}} = \mathbb{B}^\perp = \mathbb{B} \cup \{\perp_{\texttt{bool}}\}$,
$Y_{\texttt{nat}} = \mathbb{N} \cup \mathsf{bot}\,(\mathbb{N})$ with $0^\perp = \perp_{\texttt{nat}}$,
$Y_{\texttt{int}} = \mathbb{Z} \cup \{\perp_{\texttt{int}}\}$, $\quad Y_{\texttt{pair}} = Y_{\texttt{int}}^2 \cup \{\perp_{\texttt{pair}}\}$,
$Y_{\texttt{list}} = Y_{\texttt{int}}^* \cup \mathsf{bot}\,(Y_{\texttt{int}}^*)$ with $\varepsilon^\perp = \perp_{\texttt{list}}$.)

It is then straightforward to check that with these partial orders (Y, q) becomes an intrinsic ordered Λ-algebra.

Ordered algebras will be obtained as ordered extensions of bottomed algebras: If (Y, p) is a bottomed Λ-algebra then an ordered Λ-algebra (X, p) is said to be an *ordered extension* of (Y, p) if (Y, p) is the underlying bottomed algebra of (X, p). Defining an ordered extension of a given bottomed Λ-algebra (Y, p) thus amounts to specifying an appropriate family of partial orders which turns p into a family of monotone mappings. The main result of the section (Proposition 5.1.1) says that each regular bottomed Λ-algebra satisfying some additional conditions has a unique intrinsic ordered extension.

The simplest example of an ordered Λ-algebra is obtained by starting with the flat bottomed extension $(Y^\perp, q^\perp)$ of a Λ-algebra (Y, q) and considering $Y_b^\perp$ as a poset with $\sqsubseteq_b$ the flat order on $Y_b^\perp$ (thus $y \sqsubseteq_b y'$ if and only if $y \in \{\perp_b, y'\}$). Then each of the mappings $q_k^\perp$ is monotone (since if $v_1,\ v_2 \in (Y^\perp)^{k^\triangleright}$ with $v_1 \sqsubseteq^{k^\triangleright} v_1$ and $q_k^\perp(v_1) \neq \perp_{k_\triangleleft}$ then $v_1 \in Y^{k^\triangleright}$ and $v_1 = v_2$). This means that $(Y^\perp, q^\perp)$ is an ordered Λ-algebra which will be called the *flat ordered extension of* (Y, q). If (Y, p) is an initial Λ-algebra then, as was already noted, the bottomed Λ-algebra $(Y^\perp, q^\perp)$ is regular, and in this case the ordered Λ-algebra $(Y^\perp, q^\perp)$ is trivially intrinsic.

Recall that a bottomed Λ-algebra (Y, q) is said to be $\natural$-stable if whenever $k \in K$ and $v_1,\ v_2 \in Y^{k^\triangleright}$ are such that $q_k(v_1) \in Y_{k_\triangleleft}^\natural$ and $v_2(\eta) \in Y_{k^\triangleright\eta}^\natural$ for all $\eta \in \langle k^\triangleright \rangle$ with $v_1(\eta) \in Y_{k^\triangleright\eta}^\natural$ then also $q_k(v_2) \in Y_{k_\triangleleft}^\natural$. By Proposition 3.3.5 a bottomed Λ-algebra is $\natural$-stable if and only if it is a $\diamond$-algebra for some $\natural$-stable simple head type $\diamond$. Finally, recall that the simple head types $\diamond^\perp$, $\diamond^\natural$ and $\diamond^\bowtie$, as well as the degenerate head type $\diamond^\flat$, are all $\natural$-stable.

Lemma 5.1.1 *A simple head type $\diamond$ is $\natural$-stable if and only if $(H, \diamond)$ is an ordered Λ-algebra, naturally considering $\mathbb{T}$ as a poset with $\perp \sqsubseteq \natural$.*

Proof This is clear. □

As usual, we consider a set-up including the situation typical for open signatures. Let V be an A-family of posets, which is considered to be fixed in what follows. An ordered Λ-algebra (X, p) is said to be *bound to* V if $X_{|A} = V$. Of course, if Λ is closed (i.e., if $A = \varnothing$) then there is only one A-family of posets and any ordered Λ-algebra is bound to it. The underlying bottomed set of the poset V_b will be denoted by U_b. Thus U also denotes the corresponding family of bottomed sets.

Proposition 5.1.1 *Let (Y, p) be a U-minimal regular $\natural$-stable Λ-algebra. Then there exists a unique intrinsic ordered extension (X, p) of (Y, p) which is bound to V.*

Proof This occupies the second half of the section. □

If (X, p) and (Y, q) are ordered Λ-algebras then an *ordered homomorphism* π from (X, p) to (Y, q) is a bottomed homomorphism of the underlying bottomed algebras such that π_b is a monotone mapping for each $b \in B$. If (X, p) and (X', p') are ordered Λ-algebras bound to V then a an ordered homomorphism $\pi : (X, p) \to (X', p')$ is said to *fix* V if $\pi_a(x) = x$ for each $x \in V_a$, $a \in A$. Again, if Λ is closed then this imposes no requirement on an ordered homomorphism.

Proposition 5.1.2 *(1) If (X, p) is an ordered Λ-algebra bound to V then the B-family of identity mappings* id $: X \to X$ *defines an ordered homomorphism from (X, p) to itself fixing V.*

(2) If $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are ordered homomorphisms fixing V then the composition $\varrho \circ \pi$ is an ordered homomorphism from (X, p) to (Z, r) fixing V.

Proof This follows immediately from Propositions 2.2.1 and 4.1.1. $\square$

Proposition 5.1.2 implies there is a category whose objects are ordered Λ-algebras bound to V with morphisms ordered homomorphisms fixing V. There exist initial objects in this category but, as in Chapter 3, they are really too special. What is required is the analogue of Proposition 3.3.5, and this is given in Proposition 5.1.3 below.

An ordered Λ-algebra (X, p) is defined to be *V-minimal* if it is bound to V and the underlying bottomed Λ-algebra is U-minimal. Similarly, it is defined to be an *$(H, \diamond)$-algebra* if the underlying bottomed Λ-algebra is an $(H, \diamond)$-algebra.

Proposition 5.1.3 *For each $\natural$-stable simple head type $\diamond$ there exists a V-minimal intrinsic ordered $\diamond$-algebra. Moreover, any such $\diamond$-algebra (X, p) is an initial object in the full subcategory of ordered $\diamond$-algebras bound to V. In fact, (X, p) is intrinsically free: For each ordered $\diamond$-algebra (Y, q) and each family of bottomed monotone mappings $\tau : V \to Y_{|A}$ there exists a unique ordered homomorphism $\pi : (X, p) \to (Y, q)$ such that $\pi_{|A} = \tau$.*

Proof Proposition 3.3.1 implies there exists a U-minimal regular $\diamond$-algebra (Y, p), and by Proposition 3.3.5 (Y, q) is $\natural$-stable. Thus by Proposition 5.1.1 there exists an intrinsic ordered extension (X, p) of (Y, p) bound to V. But this means that (X, p) is a V-minimal intrinsic ordered $\diamond$-algebra. To show that any such $\diamond$-algebra is intrinsically free the following fact is needed:

Lemma 5.1.2 *Let (X, p) be a V-minimal intrinsic ordered Λ-algebra, let (Y, q) be an ordered Λ-algebra and $\pi : (X, p) \to (Y, q)$ a bottomed homomorphism of the underlying bottomed Λ-algebras such that the mapping π_a is monotone for each $a \in A$. Then π_b is monotone for each $b \in B$, i.e., π is ordered.*

Proof For each $b \in B$ let X'_b denote the set of those elements $x \in X_b$ such that $\pi_b(x) \sqsubseteq_b \pi_b(x')$ for all $x' \in X_b$ with $x \sqsubseteq_b x'$. Then $X'_{|A} = V$, since by assumption π_a is monotone for each $a \in A$, and $\bot_b \in X'_b$ for each $b \in B$, since $\pi_b(\bot_b) = \bot_b$. Moreover, the family X' is invariant in (X, p): Let $k \in K$ and $v \in (X')^{k^\triangleright}$, $x' \in X_{k_\triangleleft}$ with $p_k(v) \sqsubseteq_{k_\triangleleft} x'$. If $p_k(v) = \bot_{k_\triangleleft}$ then trivially $\pi_{k_\triangleleft}(p_k(v)) \sqsubseteq_{k_\triangleleft} \pi(x')$, hence suppose $p_k(v) \neq \bot_{k_\triangleleft}$. There then exists a unique $v' \in X^{k^\triangleright}$ such that $x' = p_k(v')$, and $v \sqsubseteq^{k^\triangleright} v'$. But $v(\eta) \sqsubseteq_{k^\triangleright\eta} v'(\eta)$ and $v(\eta) \in X'_{k^\triangleright\eta}$, and so $\pi_{k^\triangleright\eta}(v(\eta)) \sqsubseteq_{k^\triangleright\eta} \pi_{k^\triangleright\eta}(v'(\eta))$ for each $\eta \in \langle k^\triangleright \rangle$, i.e., $\pi^{k^\triangleright}(v) \sqsubseteq^{k^\triangleright} \pi^{k^\triangleright}(v')$. Therefore

$$\pi_{k_\triangleleft}(p_k(v)) = q_k(\pi^{k^\triangleright}(v)) \sqsubseteq_{k_\triangleleft} q_k(\pi^{k^\triangleright}(v')) = \pi_{k_\triangleleft}(p_k(v')) = \pi_{k_\triangleleft}(x') \ .$$

It follows that $X' = X$, since (X, p) is V-minimal, which implies π is a family of monotone mappings, i.e., π is an ordered homomorphism. □

Now for the proof of Proposition 5.1.1. Let (X, p) be a V-minimal intrinsic ordered $\diamond$-algebra, let (Y, q) be an ordered $\diamond$-algebra and let $\tau : V \to Y_{|A}$ be a family of bottomed monotone mappings. The underlying bottomed Λ-algebra of (X, p) is then a U-minimal regular $\diamond$-algebra, and so by Proposition 3.3.6 there exists a unique bottomed homomorphism $\pi : (X, p) \to (Y, q)$ such that $\pi_{|A} = \tau$. Thus by Lemma 5.1.2 π is an ordered homomorphism from (X, p) to (Y, q). Finally, π is unique, since any ordered homomorphism from (X, p) to (Y, q) is also a bottomed homomorphism of the underlying bottomed Λ-algebras, and by Proposition 3.3.6 there is a unique such bottomed homomorphism π with $\pi_{|A} = \tau$. □

Before starting with the proof of Proposition 5.1.1 we first look at a couple of elementary properties of intrinsic ordered Λ-algebras. A poset X is said to be *locally finite* if the principal ideal $\{y \in X : y \sqsubseteq x\}$ is finite for each $x \in X$. (For a partially ordered set with a bottom element this is equivalent to the usual definition of being locally finite, which is that $\{y \in X : x' \sqsubseteq y \sqsubseteq x\}$ should be finite for all x, $x' \in X$.)

Proposition 5.1.4 *If (X, p) is a V-minimal intrinsic ordered Λ-algebra and the posets in the A-family V are all locally finite then X is a B-family of locally finite posets.*

Proof For each $b \in B$ let $X^\diamond_b$ denote the set of elements $x \in X_b$ for which the ideal $\{y \in X : y \sqsubseteq x\}$ is finite. Then clearly $\bot_b \in X^\diamond_b$ and it it follows more-or-less directly from the definition of being intrinsic that $X^\diamond$ is an invariant family in the underlying bottomed Λ-algebra. Moreover, by assumption $X^\diamond_{|A} = V$, and hence $X^\diamond = X$, since (X, p) is V-minimal, i.e., X is a B-family of locally finite posets. □

Note the following special case of Proposition 5.1.4: If the signature Λ is closed (i.e., if $A = \varnothing$) and (X, p) is a minimal intrinsic ordered Λ-algebra then X is always a B-family of locally finite posets.

If X is a poset then $x \in X$ is said to be *maximal* if $\{y \in X : x \sqsubseteq y\} = \{x\}$; the set of maximal elements will be denoted by $X^{\uparrow}$.

Lemma 5.1.3 *If (X, p) is an intrinsic ordered Λ-algebra then $X^{\uparrow}$ is an invariant family in the underlying bottomed Λ-algebra.*

Proof This follows more-or-less directly from the definition of being intrinsic. □

We now prepare for the proof of Proposition 5.1.1; this involves defining a family of partial orders having certain properties, and these properties form the basis for the following definition: Let (Y, p) be a regular bottomed Λ-algebra, and for each $b \in B$ let $\sqsubseteq_b$ be a partial order on the set Y_b. Then the B-family of partial orders $\sqsubseteq$ will be called an *ordering associated* with (Y, p) (or just an *associated ordering*) if the following two conditions hold:

(1) $(Y_b, \sqsubseteq_b)$ is a poset with bottom element $\perp_b$ for each $b \in B$.

(2) If $b \in B \setminus A$ and y_1, $y_2 \in Y_b^{\natural}$ then $y_1 \sqsubseteq_b y_2$ if and only if $y_1 = p_k(v_1)$ and $y_2 = p_k(v_2)$ with $k \in K_b$ and $v_1 \sqsubseteq^{k^{\triangleright}} v_2$.

If (X, p) is an intrinsic ordered extension of (Y, p) and $\sqsubseteq_b$ is the partial order on X_b for each $b \in B$ then $\sqsubseteq$ is an ordering associated with (Y, p). Conversely, if $\sqsubseteq$ is an ordering associated with (Y, p) and $X_b = (Y_b, \sqsubseteq_b)$ for each $b \in B$ then (X, p) will be an intrinsic ordered extension of (Y, p), provided (X, p) is an ordered Λ-algebra. But in general this is not the case, since it is possible that there exist v_1, $v_2 \in X^{k^{\triangleright}}$ with $v_1 \sqsubseteq^{k^{\triangleright}} v_2$ and $p_k(v_1) \neq \perp_{k_{\triangleleft}}$ but with $p_k(v_2) = \perp_{k_{\triangleleft}}$ (see Example 5.1.4 at the end of the section). However, if (Y, q) is $\natural$-stable then this problem does not arise:

Lemma 5.1.4 *Let (Y, p) be a $\natural$-stable regular bottomed Λ-algebra, let $\sqsubseteq$ be an ordering associated with (Y, p) and put $X_b = (Y_b, \sqsubseteq_b)$ for each $b \in B$. Then (X, p) is an ordered Λ-algebra (and thus an intrinsic ordered extension of (Y, p)).*

Proof It must be shown that the mappings in the family p are monotone. Thus consider $k \in K$ and let v_1, $v_2 \in X^{k^{\triangleright}}$ with $v_1 \sqsubseteq^{k^{\triangleright}} v_2$; if $p_k(v_1) = \perp_{k_{\triangleleft}}$ then of course $p_k(v_1) \sqsubseteq_{k_{\triangleleft}} p_k(v_2)$ holds trivially, and so it can be assumed that $p_k(v_1) \in Y^{\natural}_{k_{\triangleleft}}$. Now $v_2(\eta) \in Y^{\natural}_{k^{\triangleright}\eta}$ for all $\eta \in \langle k^{\triangleright} \rangle$ with $v_1(\eta) \in \perp_{k^{\triangleright}\eta}$, since $v_1(\eta) \sqsubseteq_{k^{\triangleright}\eta} v_2(\eta)$ for each $\eta \in \langle k^{\triangleright} \rangle$, and therefore $p_k(v_2) \in Y^{\natural}_{k_{\triangleleft}}$ (since (Y, p) is $\natural$-stable). Hence by the

definition of an associated ordering $p_k(v_1) \sqsubseteq_{k_\triangleleft} p_k(v_2)$, and this shows that p_k is monotone. $\square$

Example 5.1.2 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 2.2.3, let V be an A-family of posets and let (Y, q) be the bottomed Λ-algebra defined in Example 3.1.3 bound to the underlying sets in the family V. For each $b \in B \setminus A$ define a partial order $\sqsubseteq_b$ on Y_b as follows:

If $b \in \{\mathtt{bool}, \mathtt{atom}, \mathtt{int}\}$ and a, $a' \in Y_b$ then $a \sqsubseteq_b a'$ if and only if $a \in \{\perp_b, a'\}$.

If p, $p' \in Y_{\mathtt{pair}}$ then $p \sqsubseteq_{\mathtt{pair}} p'$ if and only if either $p = \perp_{\mathtt{pair}}$ or $p = (x, y)$ and $p' = (x', y')$ with x, $x' \in V^\natural_{\mathtt{x}}$, y, $y' \in V^\natural_{\mathtt{y}}$, $x \sqsubseteq_{\mathtt{x}} x'$ and $y \sqsubseteq_{\mathtt{y}} y'$.

If ℓ, $\ell' \in Y_{\mathtt{list}}$ then $\ell \sqsubseteq_{\mathtt{list}} \ell'$ if and only if either $\ell = \perp_{\mathtt{list}}$ or ℓ, $\ell' \in (Y^\natural_{\mathtt{z}})^*$ with $\ell = z_1 \cdots z_m$, $\ell' = z'_1 \cdots z'_m$ and $z_k \sqsubseteq_{\mathtt{z}} z'_k$ for each $k = 1, \ldots, m$.

If y, $y' \in Y_{\mathtt{lp}}$ then $y \sqsubseteq_{\mathtt{lp}} y'$ if and only if either $y = \perp_{\mathtt{lp}}$ or y, $y' \in Y^\natural_{\mathtt{pair}}$ with $y \sqsubseteq_{\mathtt{pair}} y'$ or y, $y' \in Y^\natural_{\mathtt{list}}$ with $y \sqsubseteq_{\mathtt{list}} y'$.

(Recall here that

$$Y_{\mathtt{bool}} = \mathbb{B}^\perp = \mathbb{B} \cup \{\perp_{\mathtt{bool}}\}, \quad Y_{\mathtt{atom}} = \mathbb{I}^\perp = \mathbb{I} \cup \{\perp_{\mathtt{atom}}\},$$
$$Y_{\mathtt{int}} = \mathbb{Z}^\perp = \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}, \quad Y_{\mathtt{pair}} = (V^\natural_{\mathtt{x}} \times V^\natural_{\mathtt{y}}) \cup \{\perp_{\mathtt{pair}}\},$$
$$Y_{\mathtt{list}} = (V^\natural_{\mathtt{z}})^* \cup \{\perp_{\mathtt{list}}\}, \quad Y_{\mathtt{lp}} = Y^\natural_{\mathtt{pair}} \times Y^\natural_{\mathtt{list}} \cup \{\perp_{\mathtt{lp}}\}.)$$

The reader is left to check that with these partial orders (Y, q) becomes an intrinsic ordered Λ-algebra. Thus in fact (Y, q) is a V-minimal intrinsic ordered $\diamond^\perp$-algebra.

Lemma 5.1.4 and Proposition 3.3.5 imply that Proposition 5.1.1 is an immediate corollary of the following result:

Proposition 5.1.5 *Let (Y, p) be a regular bottomed Λ-algebra bound to U and let $\leq$ be an A-family of bottomed partial orders on U, i.e., $\leq_a$ is a bottomed partial order on U_a for each $a \in A$. Then there exists an ordering $\sqsubseteq$ associated with (Y, p) such that $\sqsubseteq_{|A} = \leq$. Moreover, if (Y, p) is, in addition, U-minimal then this is the unique such associated ordering.*

Proof For the duration of this proof a B-family of partial orders $\sqsubseteq$ will be called an *extension of* $\leq$ if $\sqsubseteq_{|A} = \leq$ and $\sqsubseteq_b$ is a bottomed partial order on Y_b for each $b \in B \setminus A$. If $\sqsubseteq$ is an extension of $\leq$ then a binary relation $\sqsubseteq'_b$ can be defined on Y_b for each $b \in B$ as follows:

(1) If $a \in A$ then $\sqsubseteq'_a = \leq_a$.

(2) If $b \in B \setminus A$ and $y_1, y_2 \in Y_b^\natural$ then $y_1 \sqsubseteq'_b y_2$ holds if and only if $k_1 = k_2$ and $v_1 \sqsubseteq^{k_1^\triangleright} v_2$, where $k_1, k_2 \in K_b$ and $v_1 \in X^{k_1^\triangleright}$, $v_2 \in X^{k_2^\triangleright}$ are the unique elements such that $y_1 = p_{k_1}(v_1)$ and $y_2 = p_{k_2}(v_2)$.

(3) If $b \in B \setminus A$ then $\perp_b \sqsubseteq'_b y$ holds for all $y \in Y_b$ but $y \sqsubseteq'_b \perp_b$ does not hold for any $y \in Y_b^\natural$.

Lemma 5.1.5 *$\sqsubseteq'$ is an extension of $\leq$.*

Proof It is enough to show that $\sqsubseteq'_b$ is a partial order for each $b \in B \setminus A$, and it is clear that $\sqsubseteq'_b$ is reflexive. To show that $\sqsubseteq'_b$ is anti-symmetric consider $y_1, y_2 \in Y_b$ with $y_1 \sqsubseteq'_b y_2$ and $y_2 \sqsubseteq'_b y_1$. Then by (3) either $y_1 = y_2 = \perp_b$, or y_1 and y_2 both lie in $Y_b^\natural$, in which case $y_1 = y_2$ holds by (2) and the fact that $\sqsubseteq^{k^\triangleright}$ is anti-symmetric for each $k \in K_b$. A similar argument shows that $\sqsubseteq'_b$ is transitive. □

The family $\sqsubseteq'$ of partial orders given by Lemma 5.1.5 will be called the *first refinement* of the family $\sqsubseteq$. Now in order to show the existence of an associated ordering it is useful to first introduce a somewhat weaker concept: A B-family $\sqsubseteq$ is called a *weak ordering* if it is an extension of $\leq$ and if whenever $b \in B \setminus A$ and $y_1, y_2 \in Y_b^\natural$ are such that $y_1 \sqsubseteq_b y_2$ then $y_1 = p_k(v_1)$ and $y_2 = p_k(v_2)$ with $k \in K_b$ and $v_1 \sqsubseteq^{k^\triangleright} v_2$.

Lemma 5.1.6 *Suppose $\sqsubseteq$ is a weak ordering. Then the first refinement $\sqsubseteq'$ is also a weak ordering and $y_1 \sqsubseteq'_b y_2$ whenever $y_1 \sqsubseteq_b y_2$.*

Proof It will be shown first that if $y_1, y_2 \in Y_b$ with $y_1 \sqsubseteq_b y_2$ then $y_1 \sqsubseteq'_b y_2$. If $b \in A$ then this holds by definition, and if $b \in B \setminus A$ then the only non-trivial case is with $y_1, y_2 \in Y_b^\natural$. Here let $k_1, k_2 \in K_b$ and $v_1 \in X^{k_1^\triangleright}$, $v_2 \in X^{k_2^\triangleright}$ be the unique elements such that $y_1 = p_{k_1}(v_1)$ and $y_2 = p_{k_1}(v_2)$. Thus $p_{k_1}(v_1) \sqsubseteq_b p_{k_1}(v_2)$ and hence $k_1 = k_2$ and $v_1 \sqsubseteq^{k_1^\triangleright} v_2$ (since $\sqsubseteq$ is a weak ordering). But then $y_1 \sqsubseteq'_b y_2$ holds by the definition of $\sqsubseteq'_b$.

It remains to show that $\sqsubseteq'$ is a weak ordering, and by Lemma 5.1.5 $\sqsubseteq'$ is an extension of $\leq$. Therefore consider $b \in B \setminus A$ and $y_1, y_2 \in Y_b^\natural$ with $y_1 \sqsubseteq'_b y_2$ and let $k_1, k_2 \in K_b$ and $v_1 \in X^{k_1^\triangleright}$, $v_2 \in X^{k_2^\triangleright}$ be the unique elements such that

$y_1 = p_{k_1}(v_1)$ and $y_2 = p_{k_2}(v_2)$. Then $p_{k_1}(v_1) \sqsubseteq'_b p_{k_2}(v_2)$ and so (by the definition of $\sqsubseteq'_b$) $k_1 = k_2$ and $v_1 \sqsubseteq^{k_1^\triangleright} v_2$; hence by the first part $v_1\,(\sqsubseteq')^{k_1^\triangleright}\,v_2$. □

For each $b \in B \setminus A$ let $\sqsubseteq^0_b$ be the flat partial order on Y_b and for each $a \in A$ let $\sqsubseteq^0_a = \leq_a$ (so $\sqsubseteq^0$ can be thought of as the flat extension of $\leq$). It is clear that $\sqsubseteq^0$ is a weak ordering, and therefore by Lemma 5.1.6 a sequence of weak orderings $\sqsubseteq^m$, $m \geq 0$, can be defined by letting $\sqsubseteq^{m+1}$ be the first refinement of $\sqsubseteq^m$ for each $m \geq 0$. Now define a partial order $\sqsubseteq_b$ on Y_b for each $b \in B$ by stipulating that $y_1 \sqsubseteq_b y_2$ if and only if $y_1 \sqsubseteq^m_b y_2$ for some (and thus for all sufficiently large) $m \geq 0$. It is easy to see that $\sqsubseteq$ is then a weak ordering, and so in particular $\sqsubseteq_{|A} = \leq$.

In fact $\sqsubseteq$ is an associated ordering: Consider $v_1, v_2 \in X^{k^\triangleright}$ with $v_1 \sqsubseteq^{k^\triangleright} v_2$ and $p_k(v_1), p_k(v_2) \in Y^\natural_b$. Then $v_1(\eta) \sqsubseteq_{k^\triangleright\eta} v_2(\eta)$ for each $\eta \in \langle k^\triangleright \rangle$, and so for each η there exists $m_\eta \geq 0$ such that $v_1(\eta) \sqsubseteq^{m_\eta}_{k^\triangleright\eta} v_2(\eta)$. Put $m = \max\{m_\eta : \eta \in \langle k^\triangleright \rangle\}$; then $v_1(\eta) \sqsubseteq^m_{k^\triangleright\eta} v_2(\eta)$ for each η, and this means that $v_1 (\sqsubseteq^m)^{k^\triangleright} v_2$. Therefore $p_k(v_1) \sqsubseteq^{m+1}_b p_k(v_2)$ by the definition of $\sqsubseteq^{m+1}_b$, and hence $p_k(v_1) \sqsubseteq_b p_k(v_2)$.

The uniqueness when (Y, p) is U-minimal still has to be considered. Thus let $\sqsubseteq$ and $\sqsubseteq'$ be two orderings associated with (Y, p) with $\sqsubseteq_{|A} = \leq = \sqsubseteq'_{|A}$. For each $y \in Y_b$ define $L_b(y) = \{y' \in Y_b : y' \sqsubseteq_b y\}$ and $L'_b(y) = \{y' \in Y_b : y' \sqsubseteq'_b y\}$; put $Y'_b = \{y \in Y_b : L'_b(y) = L_b(y)\}$. Then Y' is a bottomed family with $Y'_{|A} = Y_{|A}$ and it is straightforward to check that the family Y' is invariant in (Y, p). Hence if (Y, p) is U-minimal then $Y' = Y$, i.e., $\sqsubseteq' = \sqsubseteq$. □

Proof of Proposition 5.1.1 This now follows immediately from Lemma 5.1.4 and Propositions 3.3.5 and 5.1.5. □

Finally, consider the case when Λ is the disjoint union of the signatures Λ_i, $i \in F$.

Proposition 5.1.6 *For each $i \in F$ let (X^i, p^i) be an intrinsic ordered Λ_i-algebra. Then the sum $\oplus_{i \in F}(X^i, p^i)$ is an intrinsic ordered Λ-algebra.*

Proof Straightforward. □

Example 5.1.3 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 2.2.3, let V be an A-family of posets and let (Y, q) be the bottomed Λ-algebra defined in Example 3.1.3 bound to the underlying sets in the family V. For each $b \in B \setminus A$ define a partial order $\sqsubseteq_b$ on Y_b as follows:

If $b \in \{\mathtt{bool}, \mathtt{atom}, \mathtt{int}\}$ and a, $a' \in Y_b$ then $a \sqsubseteq_b a'$ if and only if $a \in \{\perp_b, a'\}$.

If p, $p' \in Y_{\mathtt{pair}}$ then $p \sqsubseteq_{\mathtt{pair}} p'$ if and only if either $p = \perp_{\mathtt{pair}}$ or $p = (x, y)$ and $p' = (x', y')$ with x, $x' \in V_{\mathtt{x}}$, y, $y' \in V_{\mathtt{y}}$, $x \sqsubseteq_{\mathtt{x}} x'$ and $y \sqsubseteq_{\mathtt{y}} y'$.

If ℓ, $\ell' \in Y_{\mathtt{list}}$ then $\ell \sqsubseteq_{\mathtt{list}} \ell'$ if and only if either
$\ell = z_1 \cdots z_m$, $\ell' = z'_1 \cdots z'_m$ with $z_j \sqsubseteq_{\mathtt{z}} z'_j$ for each $j = 1, \ldots, m$, or
$\ell = (z_1 \cdots z_m)^\perp$ and ℓ' either $z'_1 \cdots z'_n$ or $(z'_1 \cdots z'_n)^\perp$, with $m \le n$ and $z_j \sqsubseteq_{\mathtt{z}} z'_j$ for each $j = 1, \ldots, m$.

If y, $y' \in Y_{\mathtt{lp}}$ then $y \sqsubseteq_{\mathtt{lp}} y'$ if and only if either $y = \perp_{\mathtt{lp}}$ or y, $y' \in Y_{\mathtt{pair}}$ with $y \sqsubseteq_{\mathtt{pair}} y'$ or y, $y' \in Y_{\mathtt{list}}$ with $y \sqsubseteq_{\mathtt{list}} y'$.

(Recall here that
$Y_{\mathtt{bool}} = \mathbb{B}^\perp = \mathbb{B} \cup \{\perp_{\mathtt{bool}}\}$, $Y_{\mathtt{atom}} = \mathbb{I}^\perp = \mathbb{I} \cup \{\perp_{\mathtt{atom}}\}$,
$Y_{\mathtt{int}} = \mathbb{Z}^\perp = \mathbb{Z} \cup \{\perp_{\mathtt{int}}\}$, $Y_{\mathtt{pair}} = (V_{\mathtt{x}} \times V_{\mathtt{y}}) \cup \{\perp_{\mathtt{pair}}\}$,
$Y_{\mathtt{list}} = V_{\mathtt{z}}^* \cup \mathsf{bot}\,(V_{\mathtt{z}}^*)$ with $\perp_{\mathtt{list}} = \varepsilon^\perp$.
$Y_{\mathtt{lp}} = Y_{\mathtt{pair}} \cup Y_{\mathtt{list}} \cup \{\perp_{\mathtt{lp}}\}$.)

The reader is left to check that with these partial orders (Y, q) becomes an intrinsic ordered Λ-algebra. Thus in fact (Y, q) is a V-minimal intrinsic ordered $\diamond^\natural$-algebra.

With a minor modification the above discussion also deals with the bottomed Λ-algebra (Y, q) defined in Example 3.3.2. The only difference in the bottomed Λ-algebras is that $Y_{\mathtt{pair}}$ is now taken to be $V_{\mathtt{x}} \times V_{\mathtt{y}}$ (with $\perp_{\mathtt{pair}} = (\perp_{\mathtt{x}}, \perp_{\mathtt{y}})$). The order $\sqsubseteq_{\mathtt{pair}}$ on $Y_{\mathtt{pair}}$ is then defined by:

If p, $p' \in Y_{\mathtt{pair}}$ then $p \sqsubseteq_{\mathtt{pair}} p'$ if and only if $p = (x, y)$ and $p' = (x', y')$ with $x \sqsubseteq_{\mathtt{x}} x'$ and $y \sqsubseteq_{\mathtt{y}} y'$.

With these partial orders (the remaining partial orders being defined as above) (Y, q) becomes a V-minimal intrinsic ordered $\diamond^\bowtie$-algebra.

Example 5.1.4 Consider the Λ-algebra (X,p) in Example 2.2.1, and let (Y,q) be the Λ-algebra defined by

$Y_{\mathtt{nat}} = \mathbb{N} \cup \{\bot_{\mathtt{nat}}, \bot^o_{\mathtt{nat}}\}$ with $\bot^o_{\mathtt{nat}} \notin \mathbb{N} \cup \{\bot_{\mathtt{nat}}\}$,
$Y_\beta = X^\bot_b$ for all $b \in B \setminus \{\mathtt{nat}\}$,
$q_{\mathtt{Zero}} : \mathbb{I} \to Y_{\mathtt{nat}}$ with $q_{\mathtt{Zero}}(\varepsilon) = 0$,

$$q_{\mathtt{Succ}} : Y_{\mathtt{nat}} \to Y_{\mathtt{nat}} \text{ with } q_{\mathtt{Succ}}(n) = \begin{cases} n+1 & \text{if } n \in \mathbb{N}, \\ \bot_{\mathtt{nat}} & \text{if } n = \bot^o_{\mathtt{nat}}, \\ \bot^o_{\mathtt{nat}} & \text{if } n = \bot_{\mathtt{nat}} \end{cases}$$

and with $q_\kappa = p^\bot_k$ for all $k \in K \setminus \{\mathtt{Zero}, \mathtt{Succ}\}$,

where $(X^\bot, p^\bot)$ is the flat bottomed extension of (X,p). Then it is easy to see that (Y,q) is a minimal regular bottomed Λ-algebra, and so by Proposition 5.1.5 there is a unique ordering $\sqsubseteq$ associated with (Y,q). However, $q_{\mathtt{Succ}}$ is not monotone, since $\bot_{\mathtt{nat}} \sqsubseteq_{\mathtt{nat}} \bot^o_{\mathtt{nat}}$ but $q_{\mathtt{Succ}}(\bot_{\mathtt{nat}}) = \bot^o_{\mathtt{nat}}$ and $q_{\mathtt{Succ}}(\bot^o_{\mathtt{nat}}) = \bot_{\mathtt{nat}}$.

5.2 Continuous algebras

A *continuous* Λ*-algebra* is any pair (X,p) in which X is a B-family of complete posets and p is a K-family of mappings such that p_k is a continuous mapping from $X^{k^\triangleright}$ to $X_{k_\triangleleft}$ for each $k \in K$, recalling that $X^{k^\triangleright}$ denotes the complete poset $\otimes(X \circ k^\triangleright)$. A continuous Λ-algebra is in particular an ordered Λ-algebra.

A continuous Λ-algebra is deemed to have a property (such as being intrinsic or being an $(H,\diamond)$-algebra) if it has this property as an ordered Λ-algebra.

The main result here is Proposition 5.2.2, which is the result corresponding to Proposition 5.1.3 for continuous algebras. The continuous Λ-algebras we will be dealing with are all initial completions of ordered Λ-algebras, and so we must first say precisely what this means.

Lemma 5.2.1 *Let (Y,q) be an ordered Λ-algebra and for each $b \in B$ let X_b be an initial completion of the poset Y_b. Then for each $k \in K$ the mapping $q_k : Y^{k^\triangleright} \to Y_{k_\triangleleft}$ extends uniquely to a continuous mapping $p_k : X^{k^\triangleright} \to X_{k_\triangleleft}$.*

Proof By assumption the mapping q_k is monotone and so it is still monotone considered as a mapping from $Y^{k^\triangleright}$ to $X_{k_\triangleleft}$; moreover, by Proposition 4.3.3 $X^{k^\triangleright}$ is an initial completion of $Y^{k^\triangleright}$. The result therefore follows from Proposition 4.3.2 ((1) $\Rightarrow$ (2)). $\square$

Lemma 5.2.1 allows the following definition to be made: A continuous Λ-algebra (X,p) is said to be an *initial completion* of an ordered Λ-algebra (Y,q) if X_b is an initial completion of the poset Y_b for each $b \in B$ and p_k is the unique continuous extension of q_k for each $k \in K$. It follows immediately from Proposition 4.3.2 that there exists an initial completion of (Y,q).

Lemma 5.2.2 *Let (Y,q) be an intrinsic ordered Λ-algebra. Then any initial completion (X,p) of (Y,q) is also intrinsic.*

Proof Let $b \in B \setminus A$ and $x \in X_b^\natural$. Since X_b is a completion of Y_b there exists $D \in \mathrm{d}(Y_b)$ with $x = \bigsqcup D$, and then $D^\natural = D \setminus \{\perp_b\}$ is also an element of $\mathrm{d}(Y_b)$ with $x = \bigsqcup D^\natural$. If $y_1,\, y_2 \in D$ with $y_1 \sqsubseteq_b y_2$ then, since (Y,q) is intrinsic, there exists a unique $k \in K_b$ and unique elements $u_1,\, u_2 \in Y^{k^\triangleright}$ with $y_1 = q_k(u_1)$ and $y_2 = q_k(u_2)$, and then $u_1 \sqsubseteq^{k^\triangleright} u_2$. Thus, since $D^\natural$ is directed, there exists a unique $k \in K_b$ and for each $y \in D^\natural$ a unique element $u \in Y^{k^\triangleright}$ with $y = q_k(u)$. Moreover, the set $C = q_k^{-1}(D^\natural)$ is an element of $\mathrm{d}(Y^{k^\triangleright})$ and $q_k(C) = D^\natural$. Put $v = \bigsqcup C$; then, since p is continuous,

$$p_k(v) = p_k\left(\bigsqcup C\right) = \bigsqcup p_k(C) = \bigsqcup q_k(C) = \bigsqcup D^\natural = x\ .$$

This shows that for each $x \in X_b^\natural$ there exists $k \in K_b$ and an element $v \in X^{k^\triangleright}$ with $x = p_k(v)$. Now let $x_1,\, x_2 \in X_b^\natural$ with $x_1 \sqsubseteq_b x_2$ and suppose $x_1 = p_{k_1}(v_1)$ and $x_2 = p_{k_2}(v_2)$ with $k_1,\, k_2 \in K_b$ and $v_1 \in X^{k_1^\triangleright}$, $v_2 \in X^{k_2^\triangleright}$. Then, since $X^{k_i^\triangleright}$ is a completion of $Y^{k_i^\triangleright}$ for $i = 1,\, 2$, there exists $C_i \in \mathrm{d}(Y^{k_i^\triangleright})$ such that $v_i = \bigsqcup C_i$. But then

$$\begin{aligned}\bigsqcup q_{k_1}(C_1) \;&=\; \bigsqcup p_{k_1}(C_1) = p_{k_1}\left(\bigsqcup C_1\right) = p_{k_1}(v_1) = x_1\\ &\sqsubseteq_b x_2 = p_{k_2}(v_2) = p_{k_2}\left(\bigsqcup C_2\right) = \bigsqcup p_{k_2}(C_2) = \bigsqcup q_{k_2}(C_2)\ ,\end{aligned}$$

which by Proposition 4.3.2 ((1) $\Rightarrow$ (3)) and Lemma 5.2.1 implies that $q_{k_1}(C_1)$ is cofinal in $q_{k_2}(C_2)$. However, by the regularity of (Y,q) this is only possible if $k_1 = k_2$ and C_1 is cofinal in C_2, and then also $v_1 = \bigsqcup C_1 \sqsubseteq^{k_1^\triangleright} \bigsqcup C_2 = v_2$. In particular (with $x_1 = x_2 = x$) this shows, together with the first part of the proof, that for each $x \in X_b^\natural$ there exists a unique $k \in K_b$ and a unique element $v \in X^{k^\triangleright}$ with $x = p_k(v)$. Moreover, it then also clearly shows that (X,p) is intrinsic. $\square$

Lemma 5.2.3 *Let $\diamond$ be a simple head type and (Y,q) be an ordered $\diamond$-algebra. Then any initial completion (X,p) of (Y,q) is also an $\diamond$-algebra.*

Proof Let $k \in K$ and $v \in X^{k^\triangleright}$. Then there exists $C \in \mathrm{d}(Y^{k^\triangleright})$ with $v = \bigsqcup C$, thus by Lemma 4.2.2 $v(\eta) = \bigsqcup\{u(\eta) : u \in C\}$ for each $\eta \in \langle k^\triangleright\rangle$ and $p_k(v) = \bigsqcup q_k(C)$.

Therefore, since C is directed and $\langle k^\triangleright \rangle$ is finite, there exists $u \in C$ such that $p_k(v) \neq \bot_{k_\triangleleft}$ if and only if $q_k(u) \neq \bot_{k_\triangleleft}$ and such that $v(\eta) \neq \bot_{k^\triangleright \eta}$ if and only if $u(\eta) \neq \bot_{k^\triangleright \eta}$ for each $\eta \in \langle k^\triangleright \rangle$. This implies that $\varepsilon_{k_\triangleleft}(p_k(v)) = \varepsilon_{k_\triangleleft}(p_k(u))$ and that $\varepsilon_{k^\triangleright \eta}(v(\eta)) = \varepsilon_{k^\triangleright \eta}(u(\eta))$ for each $\eta \in \langle k^\triangleright \rangle$. Hence

$$\varepsilon_{k_\triangleleft}(p_k(v)) = \varepsilon_{k_\triangleleft}(p_k(v')) = \varepsilon_{k_\triangleleft}(q_k(v')) = \diamond_k(\varepsilon^{k^\triangleright}(v')) = \diamond_k(\varepsilon^{k^\triangleright}(v)) \ ,$$

which shows that (X, p) is also a $\diamond$-algebra. □

Let (X, p) and (Y, q) be continuous Λ-algebras; then a *continuous homomorphism* π from (X, p) to (Y, q) is an ordered homomorphism such that π_b is a continuous mapping for each $b \in B$.

In what follows let V be an A-family of complete posets.

Proposition 5.2.1 *(1) If (X, p) is a continuous Λ-algebra bound to V then the B-family of identity mappings* $\mathrm{id} : X \to X$ *defines a continuous homomorphism from (X, p) to itself fixing V.*

(2) If $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are continuous homomorphisms fixing V then the composition $\varrho \circ \pi$ is a continuous homomorphism from (X, p) to (Z, r) fixing V.

Proof This follows immediately from Propositions 2.2.1 and 4.2.1. □

Proposition 5.2.1 implies that there is a category whose objects are continuous Λ-algebras bound to V with morphisms continuous homomorphisms fixing V. Lemma 4.2.1 implies that a morphism π in this category is an isomorphism if and only if the mapping π_b is an order isomorphism for each $b \in B$.

Now let $\diamond$ be a $\natural$-stable simple head type and let V be an A-family of algebraic posets. For $a \in A$ let $U_a = \mathrm{K}(V_a)$ be the set of algebraic elements V_a, considered as a subposet of V_a. By Proposition 5.1.3 there then exists a U-minimal intrinsic ordered $\diamond$-algebra (Y, q). Now by Proposition 4.3.6 V_a is an initial completion of U_a for each $a \in A$, and hence there exists an initial completion (X, p) of (Y, q) with $X_a = V_a$ for each $a \in A$. Thus by Lemmas 5.2.2 and 5.2.3 (X, p) is an intrinsic continuous $\diamond$-algebra, and by definition (X, p) is bound to V. Moreover, by Proposition 4.3.6 X is a family of algebraic posets.

Proposition 5.2.2 (X, p) *is an initial object in the full subcategory of continuous $\diamond$-algebras bound to V. In fact (X, p) is intrinsically free: For each continuous $\diamond$-algebra (X', p') and each family $\tau : V \to X'_{|A}$ of bottomed continuous mappings there exists a unique continuous homomorphism $\pi : (X, p) \to (X', p')$ such that $\pi_{|A} = \tau$.*

Proof Let $\tau' : U \to X'_{|A}$ be the restriction of τ to U. Then τ' is a family of bottomed monotone mappings and hence by Proposition 5.1.3 there exists a unique ordered homomorphism $\varrho : (Y, q) \to (X', p')$ such that $\varrho_{|A} = \tau'$. Now by Proposition 4.3.2 ((1) $\Rightarrow$ (2)) the monotone mapping $\varrho_b : Y_b \to X'_b$ extends uniquely to a continuous mapping $\pi_b : X_b \to X'_b$ and π is a homomorphism from (X, p) to (X', p'): Let $k \in K$; Proposition 4.2.4 implies that $\pi^{k^\triangleright} : X^{k^\triangleright} \to (X')^{k^\triangleright}$ is continuous, and so $\pi_{k_\triangleleft} \circ p_k$ and $p'_k \circ \pi^{k^\triangleright}$ are both continuous mappings from $X^{k^\triangleright}$ to $X'_{k_\triangleleft}$ with

$$(\pi_{k_\triangleleft} \circ p_k)(v) = \pi_{k_\triangleleft}(p_k(v)) = \varrho_{k_\triangleleft}(q_k(v)) = q'_k(\varrho^{k^\triangleright})(v) = p'_k(\pi^{k^\triangleright})(v) = (p'_k \circ \pi^{k^\triangleright})(v)$$

for all $v \in Y^{k^\triangleright}$. Hence by Lemma 4.3.1 $\pi_{k_\triangleleft} \circ p_k = p'_k \circ \pi^{k^\triangleright}$, since $X^{k^\triangleright}$ is a completion of $Y^{k^\triangleright}$, i.e., $\pi : (X, p) \to (X', p')$ is a continuous homomorphism, and clearly $\pi_{|A} = \tau$, since τ_a is the unique continuous extension of τ'_a for each $a \in A$. Finally, if π and π' are continuous homomorphisms from (X, p) to (X', p') with $\pi_{|A} = \tau = \pi'_{|A}$ then by Lemma 3.1.3 $\pi_b(y) = \pi'_b(y)$ for all $y \in Y_b$, $b \in B$, and so by Lemma 4.3.1 $\pi = \pi'$. $\square$

Chapter 6

Polymorphism

This chapter deals with polymorphism as it appears in all modern functional programming languages. In order to make things simpler our approach is a bit more restrictive than that allowed in languages such as *Haskell* or *ML*. However, this does not really impose any restrictions on what one (as programmer) can actually do.

6.1 Algebras in categories

As always let $\Lambda = (B, K, \Theta)$ be a signature with parameter set A. In Chapter 2 Λ-algebras were introduced as pairs (X, p) with X a B-family of sets and p an appropriate K-family of mappings. Then in Chapter 3 bottomed Λ-algebras (X, p) were considered; the family X is then a B-family of bottomed sets. Finally, in Chapter 5 ordered and continuous Λ-algebras were introduced with a B-family of posets (resp. complete posets) and a K-family of monotone (resp. continuous) mappings.

In order to emphasise what is common in all these cases it is worth formulating what it means for an algebra to be defined in a category. The resulting formalism is here little more than abstract nonsense, but it will play a useful role in the rest of the chapter.

In what follows let C be a category (as defined in Section 2.1). The objects of C will be denoted by $\mathcal{C}$ and the class of all morphisms by $\mathcal{M}$, i.e., $\mathcal{M}$ is the union of all the sets $\mathrm{Hom}(X, Y)$, X, $Y \in \mathcal{C}$.

If S is a set and $\varphi : S \to \mathcal{M}$ is an S-family of morphisms then $\varphi : X \to Y$ will be written to indicate that X and Y are the S-families of objects such that $\varphi_s \in \mathrm{Hom}(X_s, Y_s)$ for each $s \in S$. If $\psi : Y \to Z$ a further S-family of morphisms then the S-family of composed morphisms will be denoted by $\psi \circ \varphi$,

thus $\psi \circ \varphi : X \to Z$ is the S-family of morphisms with $(\psi \circ \varphi)_s = \psi_s \circ \varphi_s$ for each $s \in S$.

If $\mathcal{S}$ is a class then the class of all finite families of elements from $\mathcal{S}$ will be denoted by $\mathcal{F}(\mathcal{S})$. Note that if S is an arbitrary set and $X : S \to \mathcal{S}$ an S-family of elements from $\mathcal{S}$ then $X \circ \gamma \in \mathcal{F}(\mathcal{S})$ for each finite S-typing $\gamma \in \mathcal{T}(S)$.

The category C together with mappings $\otimes : \mathcal{F}(\mathcal{C}) \to \mathcal{C}$ and $\otimes : \mathcal{F}(\mathcal{M}) \to \mathcal{M}$ will be called a $\otimes$-*category* if the following conditions hold:

— $\otimes\varphi \in \mathrm{Hom}(\otimes X, \otimes Y)$ whenever $\varphi \in \mathcal{F}(\mathcal{M})$ with $\varphi : X \to Y$.

— $(\otimes\varrho) \circ (\otimes\pi) = \otimes(\varrho \circ \pi)$ whenever $\pi, \varrho \in \mathcal{F}(\mathcal{M})$ with $\pi : X \to Y$ and $\varrho : Y \to Z$.

— $\otimes\mathrm{id} = \mathrm{id}_{\otimes X}$ for each $X \in \mathcal{F}(\mathcal{C})$, with $\mathrm{id} : X \to X$ the family of identity morphisms.

The object $\otimes X$ should be thought of as the product of the objects in the family X and the morphism $\otimes\varphi$ as the product of the morphisms in the family φ. However, in general no assumptions will placed on the mappings $\otimes$ to justify this interpretation.

The categories Sets, BSets, Posets, $\mathsf{CPosets}$ and $\mathsf{APosets}$ are all $\otimes$-categories: In each case $\otimes : \mathcal{F}(\mathcal{C}) \to \mathcal{C}$ is defined as in Section 2.1 or Chapter 4 and if $\varphi : X \to Y$ is a finite S-family of morphisms then $\otimes\varphi : \otimes X \to \otimes Y$ is defined by

$$\otimes\varphi(v)(s) = \varphi_s(v(s))$$

for each $v \in \otimes X$, $s \in S$, noting that in all of these cases a morphism is a mapping between the underlying sets. The conditions imposed on the mapping $\otimes : \mathcal{F}(\mathcal{M}) \to \mathcal{M}$ thus follow from Lemma 2.1.

In what follows let C be a $\otimes$-category. If S is a set and $X : S \to \mathcal{C}$ an S-family of objects then for each finite S-typing γ the object $\otimes(X \circ \gamma)$ will be denoted by X^γ. In the same way, if $\pi : S \to \mathcal{M}$ is an S-family of morphisms then the morphism $\otimes(\pi \circ \gamma)$ will be denoted by π^γ. Note that this notation agrees with that already being employed in the categories listed above.

Lemma 6.1.1 *(1) Let $X : S \to \mathcal{C}$ be an S-family of objects and* $\mathrm{id} : X \to X$ *the S-family of identity morphisms. Then* $\mathrm{id}^\gamma : X^\gamma \to X^\gamma$ *is also the identity morphism for each finite S-typing γ.*

(2) Let $X, Y, Z : S \to \mathcal{C}$ be S-families of objects and let $\varphi : X \to Y$ and $\psi : Y \to Z$ be S-families of morphisms. Then $(\psi \circ \varphi)^\gamma = \psi^\gamma \circ \varphi^\gamma$ for each finite S-typing γ.

(3) Let $X, Y : S \to \mathcal{C}$ be S-families of objects and let $\varphi : X \to Y$ be an S-family of morphisms. If $\varphi_s \in \mathrm{Hom}(X_s, Y_s)$ is an isomorphism for each $s \in S$ then for each finite S-typing γ the morphism φ^γ is an isomorphism and $(\varphi^\gamma)^{-1} = (\varphi^{-1})^\gamma$, where φ^{-1} is the S-family of morphisms with $(\varphi^{-1})_s = (\varphi_s)^{-1}$ for each $s \in S$.

Proof (1) This follows since $\mathrm{id} \circ \gamma : X^\gamma \to X^\gamma$ is a family of identity morphisms and hence $\mathrm{id}^\gamma = \otimes(\mathrm{id} \circ \gamma) = \mathrm{id}_{X^\gamma}$.

(2) It is easy to see that $(\psi \circ \varphi) \circ \gamma = (\psi \circ \gamma) \circ (\varphi \circ \gamma)$ holds (exactly as in Lemma 2.1.2) and therefore

$$(\psi \circ \varphi)^\gamma = \otimes((\psi \circ \varphi) \circ \gamma) = \otimes((\psi \circ \gamma) \circ (\varphi \circ \gamma)) = \otimes(\psi \circ \gamma) \circ \otimes(\varphi \circ \gamma) = \psi^\gamma \circ \varphi^\gamma \ .$$

(3) By (1) and (2) $(\varphi^\gamma) \circ (\varphi^{-1})^\gamma$ and $(\varphi^{-1})^\gamma \circ (\varphi^\gamma)$ are identity morphisms, hence the morphism φ^γ is an isomorphism and $(\varphi^\gamma)^{-1} = (\varphi^{-1})^\gamma$. □

Now a pair (X, p) will be called a $\Lambda(\mathsf{C})$-*algebra* if $X : B \to \mathcal{C}$ is a B-family of objects and $p : K \to \mathcal{M}$ a K-family of morphisms such that $p_k \in \mathrm{Hom}(X^{k^\triangleright}, X_{k_\triangleleft})$ for each $k \in K$. (The mapping $\otimes : \mathcal{F}(\mathcal{M}) \to \mathcal{M}$ is not involved in this definition; it is only needed in the definition of a homomorphism between $\Lambda(\mathsf{C})$-algebras to be given below.)

In particular, a $\Lambda(\mathsf{BSets})$-algebra is thus nothing but a bottomed Λ-algebra, a $\Lambda(\mathsf{Posets})$-algebra is an ordered Λ-algebra and a $\Lambda(\mathsf{CPosets})$-algebra is just a continuous Λ-algebra.

In order to deal with homomorphisms between $\Lambda(\mathsf{C})$-algebras it is necessary to introduce a restricted class of morphisms. This is because in all the cases being considered, with the exception of the original set-up in Chapter 2, the mappings occurring in homomorphisms are required to be bottomed. Thus in what follows let C' be a subcategory of C having the same objects as C. For each X, $Y \in \mathcal{C}$ the set of morphisms from X to Y in C' will be denoted by $\mathrm{Hom}'(X, Y)$, and the class of all such morphisms will be denoted by $\mathcal{M}'$. In all the categories listed above, with the exception of Sets, the morphisms in $\mathcal{M}'$ will be the bottomed morphisms in $\mathcal{M}$; in Sets there is no restriction, and so here $\mathcal{M}' = \mathcal{M}$, i.e., $\mathsf{C}' = \mathsf{C}$.

Let (X, p) and (Y, q) be $\Lambda(\mathsf{C})$-algebras. A B-family $\pi : X \to Y$ of morphisms from $\mathcal{M}'$ will be called a C'-*homomorphism* from (X, p) to (Y, q) if

$$q_k \circ \pi^{k^\triangleright} = \pi_{k_\triangleleft} \circ p_k$$

for each $k \in K$. This fact will also be expressed by saying that $\pi : (X, p) \to (Y, q)$ is a C'-homomorphism.

Proposition 6.1.1 *(1) The B-family of identity morphism* $\mathrm{id} : X \to X$ *defines a C'-homomorphism from a $\Lambda(\mathsf{C})$-algebra (X, p) to itself.*

(2) If $\pi : (X, p) \to (Y, q)$ and $\varrho : (Y, q) \to (Z, r)$ are C'-homomorphisms then the composition $\varrho \circ \pi$ is a C'-homomorphism from (X, p) to (Z, r).

(3) If $\pi : (X, p) \to (Y, q)$ is a C'-homomorphism and $\pi_b \in \mathrm{Hom}'(X_b, Y_b)$ is an isomorphism for each $b \in B$ then the B-family π^{-1} of inverse morphisms is a C'-homomorphism from (Y, q) to (X, p).

Proof (1) This follows immediately from Lemma 6.1.1 (1), since for each $k \in K$

$$p_k \circ \mathrm{id}^{k^\triangleright} = p_k \circ \mathrm{id}_{X^{k^\triangleright}} = p_k = \mathrm{id}_{k_\triangleleft} \circ p_k \ .$$

(2) Let $k \in K$; then by Lemma 6.1.1 (2)

$$r_k \circ (\varrho \circ \pi)^{k^\triangleright} = r_k \circ \varrho^{k^\triangleright} \circ \pi^{k^\triangleright} = \varrho_{k_\triangleleft} \circ q_k \circ \pi^{k^\triangleright} = \varrho_{k_\triangleleft} \circ \pi_{k_\triangleleft} \circ p_k = (\varrho \circ \pi)_{k_\triangleleft} \circ p_k$$

and hence $\varrho \circ \pi$ is a C'-homomorphism from (X, p) to (Z, r).

(3) Let $k \in K$. Then $q_k \circ \pi^{k^\triangleright} = \pi_{k_\triangleleft} \circ p_k$, and therefore by Lemma 6.1.1 (3) it follows that $p_k \circ (\pi^{-1})^{k^\triangleright} = p_k \circ (\pi^{k^\triangleright})^{-1} = \pi^{-1}_{k_\triangleleft} \circ q_k$, which implies that π^{-1} is a $\mathcal{C}'$-homomorphism from (Y, q) to (X, p). $\square$

Denote by $\Lambda(\mathsf{C})$ the class of all $\Lambda(\mathsf{C})$-algebras. Proposition 6.1.1 implies there is a category having $\Lambda(\mathsf{C})$ as objects and C'-homomorphisms between $\Lambda(\mathsf{C})$-algebras as morphisms; this category will be denoted by $\Lambda(\mathsf{C}, \mathsf{C}')$. In the situations we have been considering there is then given some full subcategory H of $\Lambda(\mathsf{C}, \mathsf{C}')$: For example, the main results in Chapters 3 and 5 involve a $\natural$-stable simple head type $\diamond$, and in this case the objects of H consist of the $\diamond$-algebras.

Now for an open signature the interest is mainly in algebras bound to a given A-family of objects: If $V : A \to \mathcal{C}$ is such an A-family then a $\Lambda(\mathsf{C})$-algebra (X, p) is said to be *bound to* V if $X_{|A} = V$. If (X, p) and (Y, q) are $\Lambda(\mathsf{C})$-algebras bound to V then a C'-homomorphism $\pi : (X, p) \to (Y, q)$ is said to *fix* V if $\pi_a = \mathrm{id}_{X_a}$ for each $a \in A$. Clearly the family of identity morphisms fixes V and the composition of two C'-homomorphisms fixing V is again a C'-homomorphism fixing V. For each A-family $V : A \to \mathcal{C}$ there is thus a category whose objects are $\Lambda(\mathsf{C})$-algebras bound to V and whose morphisms are C'-homomorphisms fixing V. In this category there is again the full subcategory whose objects are, in addition, objects of H. Note that the constructions in the previous chapters produce initial (and even intrinsically free) objects in these subcategories, and in each case it was possible to characterize what it means to be initial.

If H is a full subcategory of $\Lambda(\mathsf{C}, \mathsf{C}')$ then a $\Lambda(\mathsf{C})$-algebra is called an H-*algebra* if it is an object of H. Moreover, H is said to *possess intrinsically free objects* if for each A-family $V : A \to \mathcal{C}$ there exists an intrinsically free H-algebra (X, p) bound to V, this meaning that (X, p) is an H-algebra bound to V such that for each H-algebra (Y, q) and each A-family $\tau : V \to Y_{|A}$ of C'-morphisms there exists a unique C'-homomorphism $\pi : (X, p) \to (Y, q)$ such that $\pi_{|A} = \tau$.

We have the following three examples of a full subcategory H which possesses intrinsically free objects, in each case with $\diamond$ a $\natural$-stable simple head type and with C' defined by requiring the morphisms to be bottomed:

— $\mathsf{C} = \mathsf{BSets}$ with the objects of H the bottomed $\diamond$-algebras.

— $\mathsf{C} = \mathsf{Posets}$ with the objects of H the ordered $\diamond$-algebras.

— $\mathsf{C} = \mathsf{APosets}$ with the objects of H the continuous $\diamond$-algebras.

(These statements follow from Propositions 3.3.6, 5.1.3 and 5.2.2.)

Recall that if F is a non-empty set and $\Lambda_i = (B_i, K_i, \Theta_i)$ is a signature for each $i \in F$ then Λ is said to be the disjoint union of the signatures Λ_i, $i \in F$, if the following conditions hold:

(1) $B_i \cap B_i = \varnothing$ and $K_i \cap K_i = \varnothing$ whenever $i \neq j$.

(2) $B = \bigcup_{i \in F} B_i$ and $K = \bigcup_{i \in F} K_i$.

(3) $\Theta_i(k) = \Theta(k)$ for all $k \in K_i$, $i \in F$.

In particular, if A_i is the parameter set of Λ_i for each $i \in F$ then $A = \bigcup_{i \in F} A_i$ is the parameter set of Λ.

In what follows let Λ be the disjoint union of the signatures Λ_i, $i \in F$. If (X^i, p^i) is a $\Lambda_i(\mathsf{C})$-algebra for each $i \in F$ then a $\Lambda(\mathsf{C})$-algebra (X, p) can be defined by putting $X_b = X_b^i$ for each $b \in B_i$ and $p_k = p_k^i$ for each $k \in K_i$. (X, p) will be called the *sum* of the $\Lambda_i(\mathsf{C})$-algebras (X^i, p^i), $i \in F$, and will be denoted by $\oplus_{i \in F}(X^i, p^i)$. The converse also holds:

Lemma 6.1.2 *Let (X, p) be a $\Lambda(\mathsf{C})$-algebra and for each $i \in F$ put $X^i = X_{|B_i}$ and $p^i = p_{|K_i}$. Then (X^i, p^i) is a $\Lambda_i(\mathsf{C})$-algebra and $(X, p) = \oplus_{i \in F}(X^i, p^i)$.*

Proof Straightforward. $\square$

Lemma 6.1.3 *For each $i \in F$ let (X^i, p^i), (Y^i, q^i) be $\Lambda_i(\mathsf{C})$-algebras and let $\pi^i : X^i \to Y^i$ be a family of morphisms. Let π be the B-family of morphisms with $\pi_b = \pi_b^i$ for each $b \in B_i$. Then $\pi^i : (X^i, p^i) \to (Y^i, q^i)$ is a C'-homomorphism for each $i \in F$ if and only if $\pi : \oplus_{i \in F}(X^i, p^i) \to \oplus_{i \in F}(Y^i, q^i)$ is a C'-homomorphism.*

Proof Straightforward. $\square$

For each $i \in F$ let H_i be a full subcategory of $\Lambda_i(\mathsf{C}, \mathsf{C}')$. Then by Lemma 6.1.2 a full subcategory H of $\Lambda(\mathsf{C}, \mathsf{C}')$ can be defined by stipulating $(X, p) = \oplus_{i \in F}(X^i, p^i)$ to be an object of H if and only if (X^i, p^i) is an H_i-algebra for each $i \in F$. H will be called the *sum* of the subcategories H_i, $i \in F$, and will be denoted by $\oplus_{i \in F}\mathsf{H}_i$.

Proposition 6.1.2 *Let $V : A \to \mathcal{C}$ be a family of objects and for each $i \in F$ let (X^i, p^i) be an intrinsically free H_i-algebra bound to V^i, with $V^i : A_i \to \mathcal{C}$ the restriction of V to A_i. Then $\oplus_{i\in F}(X^i, p^i)$ is an intrinsically free H-algebra bound to V.*

Proof This follows immediately from Lemma 6.1.3. □

If H_i possesses intrinsically free objects for each $i \in F$ then Proposition 6.1.2 implies in particular that $\oplus_{i\in F}\mathsf{H}_i$ also possesses intrinsically free objects.

Consider for a moment a head type $(H, \diamond)$ for the signature Λ. By Lemma 2.2.1 $(H, \diamond) = \oplus_{i\in F}(H^i, \diamond^i)$, where $H^i = H_{|B_i}$ and $\diamond^i = \diamond_{|K_i}$ for each $i \in F$, and clearly $(H^i, \diamond^i)$ is a head type for the signature Λ_i. Moreover, $\diamond$ is a $\natural$-stable simple head type if and only if each $\diamond^i$ is a $\natural$-stable simple head type.

Lemma 6.1.4 *Let $\diamond$ be a $\natural$-stable simple head type.*

(1) If $\mathsf{C} = \mathsf{BSets}$ and the objects of H_i are the bottomed $\diamond^i$-algebras for each $i \in F$ then the objects of $\oplus_{i\in F}\mathsf{H}_i$ are the bottomed $\diamond$-algebras.

(2) If $\mathsf{C} = \mathsf{Posets}$ and the objects of H_i are the ordered $\diamond^i$-algebras for each $i \in F$ then the objects of $\oplus_{i\in F}\mathsf{H}_i$ are the ordered $\diamond$-algebras.

(3) If $\mathsf{C} = \mathsf{APosets}$ and the objects of H_i are the continuous $\diamond^i$-algebras for each $i \in F$ then the objects of $\oplus_{i\in F}\mathsf{H}_i$ are the continuous $\diamond$-algebras.

Proof These also follow immediately from Lemma 6.1.3. □

For the rest of the chapter we work with the set-up introduced above. This means that the following are given:

— A $\otimes$-category C with objects $\mathcal{C}$ and morphisms $\mathcal{M}$.

— A subcategory C' of C having the same objects as C.

There is then the category $\Lambda(\mathsf{C}, \mathsf{C}')$ whose objects $\Lambda(\mathsf{C})$ are the $\Lambda(\mathsf{C})$-algebras and whose morphisms are the C'-homomorphisms between $\Lambda(\mathsf{C})$-algebras.

6.2 The polymorphic signature

In what follows let T be a fixed set (which could well be empty), whose elements will be referred to as *type variables*. The aim of this section is to define a new signature $\mathbf{\Lambda} = (\mathbf{B}, \mathbf{K}, \mathbf{\Theta})$ to be called the *polymorphic signature associated with* Λ *and the type variables* T. The elements in $\mathbf{B}$ are called *polymorphic types*, and those in $\mathbf{K}$ *polymorphic operator names*.

To get an idea of what this is all about consider the second representation of the signature Λ in Example 6.2.1. This corresponds more-or-less to that used in most functional programming languages in that each type in $B \setminus A$ is augmented with the parameters (i.e., the elements of A) on which it depends. A bit more precisely, this means that each type in $B \setminus A$ must contain all of the parameters occurring on the 'right-hand side' of the 'equation' specifying it.

Example 6.2.1 Recall the signature $\Lambda = (B, K, \Theta)$ with parameter set $A = \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ from Example 2.2.3. This signature can be represented (using the conventions introduced in Example 2.2.2) in the form

$$\begin{aligned}
&\mathtt{bool} ::= \mathtt{True} \mid \mathtt{False}\\
&\mathtt{atom} ::= \mathtt{Atom}\\
&\mathtt{int} ::= \cdots\ {-2} \mid -1 \mid 0 \mid 1 \mid 2\ \cdots\\
&\mathtt{pair} ::= \mathtt{Pair\ x\ y}\\
&\mathtt{list} ::= \mathtt{Nil} \mid \mathtt{Cons\ z\ list}\\
&\mathtt{lp} ::= \mathtt{L\ list} \mid \mathtt{P\ pair}
\end{aligned}$$

An augmented form of this representation, in which the types `pair`, `list` and `lp` are provided with parameters from the set A, is the following:

$$\begin{aligned}
&\mathtt{bool} ::= \mathtt{True} \mid \mathtt{False}\\
&\mathtt{atom} ::= \mathtt{Atom}\\
&\mathtt{int} ::= \cdots\ {-2} \mid -1 \mid 0 \mid 1 \mid 2\ \cdots\\
&\mathtt{pair\ x\ y} ::= \mathtt{Pair\ x\ y}\\
&\mathtt{list\ z} ::= \mathtt{Nil} \mid \mathtt{Cons\ z\ (list\ z)}\\
&\mathtt{lp\ x\ y\ z} ::= \mathtt{L\ (list\ z)} \mid \mathtt{P\ (pair\ x\ y)}
\end{aligned}$$

The process of adding parameters to the types in the general signature Λ leads to the notion of a support. Denote by $\mathcal{P}_o(A)$ the set of all finite subsets of A. A mapping $\lfloor\cdot\rfloor : B \to \mathcal{P}_o(A)$ is called a *support* for Λ if $\lfloor a \rfloor = \{a\}$ for each $a \in A$ and $\lfloor k^{\triangleright}\eta \rfloor \subset \lfloor k_{\triangleleft} \rfloor$ for all $\eta \in \langle k^{\triangleright} \rangle$ for each $k \in K$. In general it is possible that no support exists. However, this problem does not arise if A is finite, since then

the mapping $\lfloor\cdot\rfloor^* : B \to \mathcal{P}_o(A)$ with $\lfloor b\rfloor^* = A$ for each $b \in B \setminus A$ and $\lfloor a\rfloor^* = \{a\}$ for each $a \in A$ is a support (and thus the maximal support).

Lemma 6.2.1 *If there exists a support for Λ then there exists a minimal support, i.e., a support $\lfloor\cdot\rfloor^o$ such that $\lfloor b\rfloor^o \subset \lfloor b\rfloor$ for all $b \in B$ for each support $\lfloor\cdot\rfloor$.*

Proof Define $\lfloor\cdot\rfloor^o : B \to \mathcal{P}_o(A)$ by for each $b \in B$ letting

$$\lfloor b\rfloor^o = \{a \in A : a \in \lfloor b\rfloor \text{ for every support } \lfloor\cdot\rfloor\} .$$

Then $\lfloor\cdot\rfloor^o$ is clearly a support, and thus the minimal support for Λ. □

Example 6.2.2 For the signature Λ in Example 6.2.1 the minimal support is the mapping $\lfloor\cdot\rfloor : B \to \mathcal{P}_o(A)$ defined by

$\lfloor\texttt{pair}\rfloor = \{\texttt{x},\texttt{y}\}$, $\lfloor\texttt{list}\rfloor = \{\texttt{z}\}$, $\lfloor\texttt{lp}\rfloor = \{\texttt{x},\texttt{y},\texttt{z}\}$,
$\lfloor\texttt{x}\rfloor = \{\texttt{x}\}$, $\lfloor\texttt{y}\rfloor = \{\texttt{y}\}$, $\lfloor\texttt{z}\rfloor = \{\texttt{z}\}$,
$\lfloor b\rfloor = \varnothing$ for all $b \in B \setminus \{\texttt{pair},\texttt{list},\texttt{lp},\texttt{x},\texttt{y},\texttt{z}\}$.

In all practical applications A will be finite and the minimal support is almost always the natural choice to make. However, it is convenient to work first with a general support. Let $\lfloor\cdot\rfloor : B \to \mathcal{P}_o(A)$ be a support for Λ, which is considered to be fixed for the rest of the chapter. It should be emphasised that the constructions that follow all depend on the choice of $\lfloor\cdot\rfloor$.

We start by giving a somewhat informal description of the polymorphic signature $\mathbf{\Lambda} = (\mathbf{B},\mathbf{K},\mathbf{\Theta})$. For this is it is convenient to write each element $b \in B \setminus A$ in the form $b\ a_1 \cdots a_n$, where $a_1, \ldots, a_n$ is some fixed enumeration of the elements in the set $\lfloor b\rfloor$. With this device the sets $\mathbf{B}$ and $\mathbf{K}$ can be thought of as being defined by the following rules:

(1) Each type variable $t \in T$ is an element of $\mathbf{B}$.
(2) If $b\ a_1 \cdots a_n \in B \setminus A$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbf{B}$ then $b\ \mathbf{b}_1 \cdots \mathbf{b}_n$ is an element of $\mathbf{B}$.
(3) Each element of $\mathbf{B}$ can be obtained in a unique way using (1) and (2).
(4) If $k \in K$ and with $k_\triangleleft = b\ a_1 \cdots a_m$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathbf{B}$ then $k\ \mathbf{b}_1 \cdots \mathbf{b}_m$ is an element of $\mathbf{K}$.
(5) Each element of $\mathbf{K}$ can be obtained in a unique way using (4).

The mapping $\mathbf{\Theta} : \mathbf{K} \to \mathcal{T}(\mathbf{B}) \times \mathbf{B}$ is defined as follows: Let $\mathbf{k} \in \mathbf{K}$; then $\mathbf{k}$ has a unique representation of the form $k\ \mathbf{b}_1 \cdots \mathbf{b}_m$ with $k \in K$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathbf{B}$, where m is the number of elements in the set $\lfloor k_\triangleleft \rfloor$. Now define $\mathbf{k}_\triangleleft = \mathbf{\Theta}_\triangleleft(\mathbf{k})$ to be the element $k_\triangleleft\ \mathbf{b}_1 \cdots \mathbf{b}_m$ of $\mathbf{B}$ and define $\mathbf{k}^\triangleright = \mathbf{\Theta}^\triangleright(\mathbf{k})$ to be the typing with $\langle \mathbf{k}^\triangleright \rangle = \langle k^\triangleright \rangle$ and with $\mathbf{k}^\triangleright : \langle k^\triangleright \rangle \to \mathbf{B}$ given by $\mathbf{k}^\triangleright \eta = k^\triangleright \eta\ \mathbf{b}_{\eta,1} \cdots \mathbf{b}_{\eta,n_\eta}$ for each $\eta \in \langle k^\triangleright \rangle$, where $\mathbf{b}_{\eta,1} \cdots \mathbf{b}_{\eta,n_\eta}$ are those elements from $\mathbf{b}_1 \cdots \mathbf{b}_m$ which are indexed by the elements of the subset $\lfloor k^\triangleright \eta \rfloor$ of $\lfloor k_\triangleleft \rfloor$ (and note that a support is defined so that $\lfloor k^\triangleright \eta \rfloor \subset \lfloor k_\triangleleft \rfloor$ for each $\eta \in \langle k^\triangleright \rangle$). Example 6.2.3 on the following page illustrates theses definitions.

The definition of the signature $\mathbf{\Lambda}$ must now be made precise, and we first define $\mathbf{B}$. Recall that if X and Y are sets then Y^X denotes the set of all mappings from X to Y, and if $\alpha : X \to Y$ is a mapping and J a set then α^J denotes the mapping from X^J to Y^J defined by $\alpha^J(v) = \alpha \circ v$ for all $v \in X^J$.

Suppose Y is a set which is going to be a candidate for the set of polymorphic types. Then for each $b \in B \setminus A$ a mapping $q_b : Y^{\lfloor b \rfloor} \to Y$ must also be given such that $q_b(u)$ is the new type $b\ u(a_1) \cdots u(a_n)$ for each $u \in Y^{\lfloor b \rfloor}$. Such a pair (Y, q) consisting of a set Y and a corresponding $B \backslash A$-family of mappings q will be called *a set of polymorphic types*. If (Y, q) and (Y', q') are sets of polymorphic types then a mapping $\pi : Y \to Y'$ is called a *homomorphism* if $\pi \circ q_b = q'_b \circ \pi^{\lfloor b \rfloor}$ for each $b \in B \setminus A$. The identity mapping is clearly a homomorphism and the composition of two homomorphisms is again a homomorphism. A homomorphism $\pi : Y \to Y'$ is an *isomorphism* if there exists a homomorphism $\pi' : Y' \to Y$ such that $\pi' \circ \pi = \mathrm{id}_Y$ and $\pi \circ \pi' = \mathrm{id}_{Y'}$. It is easy to see that a homomorphism is an isomorphism if and only if it is a bijection.

A set of polymorphic types (Y, q) is said to be T-*free* if $T \subset Y$ and for each set of polymorphic types (Y', q') and for each mapping $w : T \to Y'$ there exists a unique homomorphism $\pi : (Y, q) \to (Y', q')$ such that $\pi(t) = w(t)$ for each $t \in T$.

Lemma 6.2.2 *There exists a T-free set of polymorphic types (Y, q). Moreover, if (Y', q') is a further T-free set of polymorphic types then there is a unique isomorphism $\pi : (Y, q) \to (Y', q')$ such that $\pi(t) = t$ for each $t \in T$.*

Proof Let Ξ be the single-sorted signature $(B \setminus A, \lfloor \cdot \rfloor)$; then there is obviously a one–to–one correspondence between sets of polymorphic types and Ξ-algebras. Moreover, a homomorphism between sets of polymorphic types is the same as a Ξ-homomorphism. From these facts it follows that a T-free Ξ-algebra (with the set T considered as a $\mathbb{I}$-family of sets in the obvious way) corresponds exactly to a T-free set of polymorphic types, and therefore Proposition 2.5.2 implies there exists a T-free set of polymorphic types. The uniqueness holds trivially from the definition of being T-free. $\square$

Example 6.2.3 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 6.2.1 considered the with the minimal support. As in Example 6.2.1 the types `pair`, `list` and `lp` will be written as `pair x y`, `list z` and `lp x y z`. The set of polymorphic types $\mathbf{B}$ is obtained by the following rules:

(1) Each type variable $t \in T$ is an element of $\mathbf{B}$,
(2) `bool`, `atom` and `int` are elements of $\mathbf{B}$,
(3) `pair` $\mathbf{b}_1$ $\mathbf{b}_2$ is an element of $\mathbf{B}$ for all $\mathbf{b}_1$, $\mathbf{b}_2 \in \mathbf{B}$.
(4) `list` $\mathbf{b}$ is an element of $\mathbf{B}$ for all $\mathbf{b} \in \mathbf{B}$.
(5) `lp` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ is an element of $\mathbf{B}$ for all $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3 \in \mathbf{B}$.
(6) Each element of $\mathbf{B}$ can be obtained in a unique way using (1), (2), (3), (4) and (5).

Let `v` be a type variable. Examples of elements of $\mathbf{B}$ are thus:

`list v`, `pair atom int`, `list (pair v bool)`,
`pair atom (list int)`, `pair (list bool) (list v)`.

The set of polymorphic operator names $\mathbf{K}$ is obtained by the rules:

(1) `True`, `False` and `Atom` are elements of $\mathbf{K}$.
(2) $\underline{n}$ is an element of $\mathbf{K}$ for each $n \in \mathbb{Z}$.
(3) `Pair` $\mathbf{b}_1$ $\mathbf{b}_2$ is an element of $\mathbf{K}$ for all $\mathbf{b}_1$, $\mathbf{b}_2 \in \mathbf{K}$.
(4) `Nil` $\mathbf{b}$ and `Cons` $\mathbf{b}$ are elements of $\mathbf{K}$ for all $\mathbf{b} \in \mathbf{K}$.
(5) `L` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$, `P` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3 \in \mathbf{K}$ for all $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3 \in \mathbf{B}$.
(6) Each element of $\mathbf{K}$ can be obtained in a unique way using (1), (2), (3), (4) and (5).

Examples of elements of $\mathbf{K}$ are:

`Nil (pair v v)`, `Cons (pair v bool)`, `Pair (list v) int`,
`Pair atom (list int)`, `L (list bool) (list v) (pair v v)`.

In the signature $\mathbf{\Lambda} = (\mathbf{B}, \mathbf{K}, \mathbf{\Theta})$ the types of the elements of $\mathbf{K}$ are:

`True` and `False` have type $\varepsilon \to$ `bool`, `Atom` has type $\varepsilon \to$ `atom`,
$\underline{n}$ has type $\varepsilon \to$ `int` for each $n \in \mathbb{Z}$.
`Pair` $\mathbf{b}_1$ $\mathbf{b}_2$ has type $\mathbf{b}_1$ $\mathbf{b}_2 \to$ `pair` $\mathbf{b}_1$ $\mathbf{b}_2$,
`Nil` $\mathbf{b}$ has type $\varepsilon \to$ `list` $\mathbf{b}$,
`Cons` $\mathbf{b}$ has type $\mathbf{b}$ (`list` $\mathbf{b}$) $\to$ `list` $\mathbf{b}$,
`L` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ has type `list` $\mathbf{b}_3 \to$ `lp` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$,
`P` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ has type `pair` $\mathbf{b}_1$ $\mathbf{b}_2 \to$ `lp` $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$].

The element `P atom (list v) (pair v int)` thus has the type

`pair atom (list v)` $\to$ `lp atom (list v) (pair v int)`.

Let us fix a T-free set of polymorphic types $(\mathbf{B}, \#)$. Because of the uniqueness (up to isomorphism) in Lemma 6.2.2 this will be referred to as *the set of polymorphic types*. The next lemma shows that $(\mathbf{B}, \#)$ does have the properties corresponding to the rules making up the informal definition.

Lemma 6.2.3 *The set of polymorphic types* $(\mathbf{B}, \#)$ *has the following properties:*

(1) $T \subset \mathbf{B}$ *and* $\Im(\#_b) \subset \mathbf{B} \setminus T$ *for each* $b \in B \setminus A$.

(2) For each $\mathbf{b} \in \mathbf{B} \setminus T$ *there exists a unique* $b \in B \setminus A$ *and a unique element* $u \in \mathbf{B}^{\lfloor b \rfloor}$ *such that* $\mathbf{b} = \#_b(u)$.

(3) The only subset of $\mathbf{B}$ *containing* T *and invariant under the mappings* $\#_b$, $b \in B \setminus A$, *is* $\mathbf{B}$ *itself.*

Proof This follows from Proposition 2.5.3 and the equivalence of T-free sets of polymorphic types and T-free Ξ-algebras. $\square$

The set $\mathbf{B}$ will certainly be non-empty if $T \neq \varnothing$ (since $T \subset \mathbf{B}$). Moreover, it will also be non-empty provided there is at least one type in $B \setminus A$ which does not depend on any parameters:

Lemma 6.2.4 *If there exists* $b \in B \setminus A$ *with* $\lfloor b \rfloor = \varnothing$ *then* $\mathbf{B} \neq \varnothing$.

Proof Here $\mathbf{B}^{\lfloor b \rfloor} = \mathbf{B}^{\varnothing} = \mathbb{I}$ and hence $\#_b(\varepsilon) \in \mathbf{B}$. $\square$

The hypothesis in Lemma 6.2.4 will be satisfied by any signature containing the type `bool` (for any natural choice of the support). Let us now assume that $\mathbf{B} \neq \varnothing$.

We next turn to the definition of $\mathbf{K}$. Suppose Z is a set which is going to be a candidate for the set of polymorphic operator names. For each $k \in K$ a mapping $r_k : \mathbf{B}^{\lfloor k_\triangleleft \rfloor} \to Z$ must then also be given such that $r_k(u)$ is the new operator name $k\ u(a_1) \cdots u(a_n)$ for each $u \in \mathbf{B}^{\lfloor b \rfloor}$. Such a pair (Z, r) consisting of a set Z and a corresponding K-family of mappings r will be called *a set of polymorphic operator names*. If (Z, r) and (Z', r') are sets of polymorphic operator names then a mapping $\varrho : Z \to Z'$ is called a *homomorphism* if $\varrho \circ r_k = r'_k$ for all $k \in K$. It is clear that the identity mapping $\mathrm{id}_Z : Z \to Z$ is a homomorphism and that the composition of two homomorphisms is again a homomorphism. A homomorphism $\varrho : Z \to Z'$ is an *isomorphism* if there exists a homomorphism $\varrho' : Z' \to Z$ such that $\varrho' \circ \varrho = \mathrm{id}_Z$ and $\varrho \circ \varrho' = \mathrm{id}_{Z'}$. It is easy to see that a homomorphism is an isomorphism if and only if it is a bijection.

A set of polymorphic operator names (Z, r) is said to be *initial* if for each set of polymorphic operator names (Z', r') there is a unique homomorphism from (Z, r) to (Z', r').

Lemma 6.2.5 *There exists an initial set of polymorphic operator names.*

Proof An initial set of polymorphic operator names (Z, r) can be constructed explicitly by just taking Z to be the disjoint union of the sets $\mathbf{B}^{\lfloor k_\triangleleft \rfloor}$, $k \in K$, and letting $r_k : \mathbf{B}^{\lfloor k_\triangleleft \rfloor} \to Z$ be the inclusion mapping for each $k \in K$. It is then easy to check that (Z, r) is initial. (The reader is left to show that, alternatively, an initial set of polymorphic operator names can be obtained via a $\mathbf{B}$-free algebra for an appropriate signature involving two types.) □

Proposition 2.1.2 implies that any two initial sets of polymorphic operator names are isomorphic.

Now fix an initial set of polymorphic operator names $(\mathbf{K}, \#')$. Because of the uniqueness (up to isomorphism) this will be referred to as *the set of polymorphic operator names*. The next lemma shows that $(\mathbf{K}, \#')$ does have the properties corresponding to the rules making up the informal definition.

Lemma 6.2.6 *For each* $\mathbf{k} \in \mathbf{K}$ *there exists a unique* $k \in K$ *and a unique element* $u \in \mathbf{B}^{\lfloor k_\triangleleft \rfloor}$ *such that* $\mathbf{k} = \#'_k(u)$.

Proof This clearly holds for the particular initial set of polymorphic operator names given in the proof of Lemma 6.2.5, and so it must also hold for any initial set of polymorphic operator names. □

It remains to define the mapping $\mathbf{\Theta} : \mathbf{K} \to \mathcal{T}(\mathbf{B}) \times \mathbf{B}$. Before doing this, however, it is convenient to introduce two new families of mappings which reorganise the information contained in the families $\#$ and $\#'$. If $C \subset D \subset A$ and $f : \mathbf{B}^C \to Z$ is a mapping then, in order to increase the legibility, we just write $f(u)$ instead of $f(u_{|C})$ for each $u \in \mathbf{B}^D$. For each $u \in \mathbf{B}^A$ let $\flat_u : B \setminus A \to \mathbf{B}$ be the mapping given by $\flat_u(b) = \#_b(u)$ for each $b \in B \setminus A$ and $\flat'_u : K \to \mathbf{K}$ the mapping given by $\flat'_u(k) = \#'_k(u)$ for each $k \in K$ (and of course $\#_b(u)$ here really means $\#_b$ applied to the restriction of u to $\lfloor b \rfloor$ and $\#'_k(u)$ means $\#'_k$ applied to the restriction of u to $\lfloor k_\triangleleft \rfloor$).

Lemma 6.2.7 *(1) If* b, $b' \in B \setminus A$ *and* u, $v \in \mathbf{B}^A$ *then* $\flat_u(b) = \flat_v(b')$ *if and only if* $b = b'$ *and* $u_{|\lfloor b \rfloor} = v_{|\lfloor b \rfloor}$*; moreover,* $\bigcup_{u \in \mathbf{B}^A} \Im(\flat_u) = \mathbf{B} \setminus T$.

(2) If k, $k' \in K$ *and* u, $v \in \mathbf{B}^A$ *then* $\flat'_u(k) = \flat'_v(k')$ *if and only if* $k = k'$ *and* $u_{|\lfloor k_\triangleleft \rfloor} = v_{|\lfloor k_\triangleleft \rfloor}$*; moreover,* $\bigcup_{u \in \mathbf{B}^A} \Im(\flat'_u) = \mathbf{K}$.

Proof This follows immediately from Lemmas 6.2.3 and 6.2.6. □

In particular, Lemma 6.2.7 implies that $\flat_u : B \setminus A \to \mathbf{B}$ and $\flat'_u : K \to \mathbf{K}$ are both injective mappings for each $u \in \mathbf{B}^A$.

For what follows it is necessary to extend the mapping $\flat_u : B \setminus A \to \mathbf{B}$ to a mapping $\flat_u : B \to \mathbf{B}$ by putting $\flat_u(a) = u(a)$ for each $a \in A$ (but note that the statements in Lemma 6.2.7 (1) no longer hold for this extended mapping).

Let $\mathcal{B}$ and $\mathcal{C}$ be classes; a mapping $\varphi : \mathcal{B}^A \to \mathcal{C}^B$ is said to be *B-compatible* with $\lfloor\cdot\rfloor$ if $\varphi(u)(b) = \varphi(v)(b)$ whenever $u(a) = v(a)$ for all $a \in \lfloor b \rfloor$. Similarly, a mapping $\psi : \mathcal{B}^A \to \mathcal{C}^K$ is said to be *K-compatible* with $\lfloor\cdot\rfloor$ if $\psi(u)(k) = \psi(v)(k)$ whenever $u(a) = v(a)$ for all $a \in \lfloor k_\triangleleft \rfloor$. The following result really just restates the fact that $(\mathbf{B}, \#)$ is an initial set of polymorphic types and $(\mathbf{K}, \#')$ is an initial set of polymorphic operator names, but using the mappings $\flat_u$, $\flat'_u$, $u \in \mathbf{B}^A$, instead of the families $\#$ and $\#'$.

Proposition 6.2.1 *Let $\mathcal{C}$ be a class.*

(1) Let $\varphi : \mathcal{C}^A \to \mathcal{C}^B$ be a mapping which is B-compatible with $\lfloor\cdot\rfloor$ and such that $\varphi(U)(a) = U(a)$ for all $U \in \mathcal{C}^A$, $a \in A$; let $\beta : T \to \mathcal{C}$. Then there exists a unique mapping $\zeta : \mathbf{B} \to \mathcal{C}$ such that $\zeta_{|T} = \beta$ and $\zeta \circ \flat_u = \varphi(\zeta \circ u)$ for all $u \in \mathbf{B}^A$.

(2) Let $\psi : \mathbf{B}^A \to \mathcal{C}^K$ be a mapping which is K-compatible with $\lfloor\cdot\rfloor$; then there exists a unique mapping $\zeta : \mathbf{K} \to \mathcal{C}$ such that $\zeta \circ \flat'_u = \psi(u)$ for all $u \in \mathbf{B}^A$.

Proof (1) For each $b \in B \setminus A$ define $\alpha_b : \mathcal{C}^{\lfloor b \rfloor} \to \mathcal{C}$ by putting $\alpha_b(U) = \varphi(U')(b)$, where $U' : A \to \mathcal{C}$ is any extension of the mapping $U : \lfloor b \rfloor \to \mathcal{C}$ to A (and $\alpha_b(U)$ does not depend on which extension is used because of the B-compatibility with $\lfloor\cdot\rfloor$). Then, except that $\mathcal{C}$ is a class, $(\mathcal{C}, \alpha)$ would be a Ξ-algebra, with Ξ the single-sorted signature $(B \setminus A, \lfloor\cdot\rfloor)$, and as noted in the proof of Lemma 6.2.2, $(\mathbf{B}, \#)$ is a T-free Ξ-algebra. Therefore by Proposition 2.5.5 there exists a unique Ξ-homomorphism $\zeta : (\mathbf{B}, \#) \to (\mathcal{C}, \alpha)$ with $\zeta_{|T} = \beta$. But ζ being a Ξ-homomorphism means that $\zeta \circ \#_b = \alpha_b \circ \zeta^{\lfloor b \rfloor}$ for all $b \in B \setminus A$. Now let $b \in B \setminus A$, $u \in \mathbf{B}^A$ and put $u' = u_{|\lfloor b \rfloor}$. Then $\zeta \circ u = \zeta^A(u)$ is an extension of $\zeta^{\lfloor b \rfloor}(u')$ to A and so $\zeta(\flat_u(b)) = \zeta(\#_b(u')) = \alpha_b(\zeta^{\lfloor b \rfloor}(u')) = \varphi(\zeta \circ u)(b)$. On the other hand, if $a \in A$ then $\zeta(\flat_u(a)) = \zeta(u(a)) = (\zeta \circ u)(a) = \varphi(\zeta \circ u)(a)$ and hence $\zeta \circ \flat_u = \varphi(\zeta \circ u)$ for all $u \in \mathbf{B}^A$. Conversely, if $\xi : \mathbf{B} \to \mathcal{C}$ is any mapping with $\xi \circ \flat_u = \varphi(\xi \circ u)$ for all $u \in \mathbf{B}^A$ then it immediately follows that $\xi \circ \#_b = \alpha_b \circ \xi^{\lfloor b \rfloor}$ for all $b \in B \setminus A$. This implies that ζ is the unique mapping with $\zeta_{|T} = \beta$ and $\zeta \circ \flat_u = \varphi(\zeta \circ u)$ for all $u \in \mathbf{B}^A$.

(2) For each $k \in K$ define $\alpha_k : \mathbf{B}^{\lfloor k_\triangleleft \rfloor} \to \mathcal{C}$ by putting $\alpha_k(u) = \psi(u')(k)$, where $u' : A \to \mathbf{B}$ is any extension of the mapping $u : \lfloor k_\triangleleft \rfloor \to \mathbf{B}$ to A (and $\alpha_k(u)$ does not depend on which extension is used because of the K-compatibility with $\lfloor\cdot\rfloor$). Then $(\mathcal{C}, \alpha)$ would be a set of polymorphic operator names if $\mathcal{C}$ were a set (and not a class), and so there exists a unique mapping $\zeta : \mathbf{K} \to \mathcal{C}$ such that $\zeta \circ \#'_k = \alpha_k$ for each $k \in K$. ($\mathcal{C}$ being a class is clearly not a problem here, since it is not a problem if the initial set of polymorphic operator names given in the proof of Lemma 6.2.5 is used.) It follows that $\zeta(\flat'_u(k)) = \zeta(\#'_k(u')) = \alpha_k(u') = \psi(u)(k)$

for all $k \in K$, $u \in \mathbf{B}^A$, where $u' = u_{|\lfloor k_\triangleleft \rfloor}$, and hence $\zeta \circ \flat'_u = \psi(u)$ for all $u \in \mathbf{B}^A$. The uniqueness follows from the fact that $\bigcup_{u \in \mathbf{B}^A} \Im(\flat'_u) = \mathbf{K}$. $\square$

For $u \in \mathbf{B}^A$ let $\flat^*_u : \mathcal{T}(B) \to \mathcal{T}(\mathbf{B})$ be the induced mapping given by $\flat^*_u(\gamma) = \flat_u \circ \gamma$ for each $\gamma \in \mathcal{T}(B)$.

Proposition 6.2.2 *There exists a unique mapping $\mathbf{\Theta}^\triangleright : \mathbf{K} \to \mathcal{T}(\mathbf{B})$ and a unique mapping $\mathbf{\Theta}_\triangleleft : \mathbf{K} \to \mathbf{B}$ such that $\mathbf{\Theta}^\triangleright \circ \flat'_u = \flat^*_u \circ \Theta^\triangleright$ and $\mathbf{\Theta}_\triangleleft \circ \flat'_u = \flat_u \circ \Theta_\triangleleft$ for all $u \in \mathbf{B}^A$.*

Proof These are both special cases of Proposition 6.2.1 (2).

To obtain the mapping $\mathbf{\Theta}^\triangleright : \mathbf{K} \to \mathcal{T}(\mathbf{B})$ consider the mapping $\psi : \mathbf{B}^A \to \mathcal{T}(\mathbf{B})^K$ given by $\psi(u)(k) = \flat_u \circ k^\triangleright$ for all $u \in \mathbf{B}^A$, $k \in K$. Let $k \in K$ and $u, v \in \mathbf{B}^A$ with $u(a) = v(a)$ for all $a \in \lfloor k_\triangleleft \rfloor$. If $\eta \in \langle k^\triangleright \rangle$ with $k^\triangleright \eta \in B \setminus A$ then by Lemma 6.2.7 (1) $\psi(u)(k)(\eta) = \flat_u(k^\triangleright \eta) = \flat_v(k^\triangleright \eta) = \psi(v)(k)(\eta)$, since $\lfloor k^\triangleright \eta \rfloor \subset \lfloor k_\triangleleft \rfloor$. On the other hand, if $k^\triangleright \eta \in A$ then

$$\psi(u)(k)(\eta) = \flat_u(k^\triangleright \eta) = u(k^\triangleright \eta) = v(k^\triangleright \eta) = \flat_v(k^\triangleright \eta) = \psi(v)(k)(\eta) \ ,$$

since $k^\triangleright \eta \in \{k^\triangleright \eta\} = \lfloor k^\triangleright \eta \rfloor \subset \lfloor k_\triangleleft \rfloor$. This shows that ψ is K-compatible with $\lfloor \cdot \rfloor$ and thus by Proposition 6.2.1 (2) there exists a unique mapping $\mathbf{\Theta}^\triangleright : \mathbf{K} \to \mathcal{T}(\mathbf{B})$ such that $\mathbf{\Theta}^\triangleright \circ \flat'_u = \psi(u) = \flat^*_u \circ \Theta^\triangleright$ for all $u \in \mathbf{B}^A$.

To obtain the mapping $\mathbf{\Theta}_\triangleleft : \mathbf{K} \to \mathbf{B}$ define $\psi : \mathbf{B}^A \to \mathbf{B}^K$ by $\psi(u)(k) = \flat_u(k_\triangleleft)$ for all $u \in \mathbf{B}^A$, $k \in K$. By Lemma 6.2.7 (1) ψ is K-compatible with $\lfloor \cdot \rfloor$ and thus by Proposition 6.2.1 (2) there exists a unique mapping $\mathbf{\Theta}_\triangleleft : \mathbf{K} \to \mathbf{B}$ such that $\mathbf{\Theta}_\triangleleft \circ \flat'_u = \psi(u) = \flat_u \circ \Theta_\triangleleft$ for all $u \in \mathbf{B}^A$. $\square$

Now put $\mathbf{\Theta} = (\mathbf{\Theta}^\triangleright, \mathbf{\Theta}_\triangleleft)$. This completes the formal definition of the polymorphic signature $\mathbf{\Lambda} = (\mathbf{B}, \mathbf{K}, \mathbf{\Theta})$. Note that if $u \in \mathbf{B}^A$, $k \in K$ and $\mathbf{k} = \flat'_u(k)$ then by definition $\mathbf{k}_\triangleleft = \flat_u(k_\triangleleft)$ and $\mathbf{k}^\triangleright = \flat^*_u(k^\triangleright) = \flat_u \circ k^\triangleright$.

Proposition 6.2.3 *(1) Let $b \in B \setminus A$ and $u \in \mathbf{B}^A$ and put $\mathbf{b} = \flat_u(b)$. Then $\mathbf{K_b} = \{\flat'_u(k) : k \in K_b\}$ (and so in particular $\mathbf{K_b} \neq \varnothing$, since $K_b \neq \varnothing$).*

(2) $\Im(\mathbf{\Theta}_\triangleleft) = \mathbf{B} \setminus T$, and thus T is the parameter set of $\mathbf{\Lambda}$.

Proof (1) Consider $\mathbf{k} \in \mathbf{K_b}$, so $\mathbf{k}_\triangleleft = \mathbf{b}$. By Lemma 6.2.7 (2) there exist $k \in K$ and $v \in \mathbf{B}^A$ with $\mathbf{k} = \flat'_v(k)$ and then $\flat_v(k_\triangleleft) = \mathbf{k}_\triangleleft = \mathbf{b} = \flat_u(b)$. Hence by Lemma 6.2.7 (1) $k_\triangleleft = b$ and $v_{|\lfloor k_\triangleleft \rfloor} = u_{|\lfloor k_\triangleleft \rfloor}$. Therefore $k \in K_b$ and by Lemma 6.2.7 (2) $\mathbf{k} = \flat'_v(k) = \flat'_u(k)$. Conversely, let $k \in K_b$ and put $\mathbf{k} = \flat'_u(k)$; then $k_\triangleleft = b$ and thus $\mathbf{k}_\triangleleft = \flat_u(k_\triangleleft) = \flat_u(b) = \mathbf{b}$, i.e., $\mathbf{k} \in \mathbf{K_b}$. This shows that $\mathbf{K_b} = \{\flat'_u(k) : k \in K_b\}$.

(2) Let $\mathbf{k} \in \mathbf{K}$; then by Lemma 6.2.7 (2) there exists $k \in K$ and $u \in \mathbf{B}^A$ such that $\mathbf{k} = \flat'_u(k)$ and thus

$$\mathbf{\Theta}_{\triangleleft}(\mathbf{k}) = \mathbf{\Theta}_{\triangleleft}(\flat'_u(k)) = \flat_u(\Theta_{\triangleleft}(k)) \in \Im(\flat_u) \subset \mathbf{B} \setminus T \ .$$

Conversely, if $\mathbf{b} \in \mathbf{B} \setminus T$ then by (1) $\mathbf{K}_\mathbf{b} \neq \varnothing$, and hence $\mathbf{b} \in \Im(\mathbf{\Theta}_{\triangleleft})$. $\square$

Proposition 6.2.4 *Let* $\mathbf{X} : \mathbf{B} \to \mathcal{C}$ *be a* $\mathbf{B}$*-family of objects and* $\mathbf{p} : \mathbf{K} \to \mathcal{M}$ *a* $\mathbf{K}$*-family of morphisms. Then the pair* $(\mathbf{X}, \mathbf{p})$ *is a* $\mathbf{\Lambda}(\mathsf{C})$*-algebra if and only if* $(\mathbf{X} \circ \flat_u, \mathbf{p} \circ \flat'_u)$ *is a* $\Lambda(\mathsf{C})$*-algebra for each* $u \in \mathbf{B}^A$*.*

Proof Let $k \in K$, $u \in \mathbf{B}^A$ and put $\mathbf{k} = \flat'_u(k)$. Then $\mathbf{k}_{\triangleleft} = \flat_u(k_{\triangleleft})$, $\mathbf{k}^{\triangleright} = \flat_u \circ k^{\triangleright}$ and thus $\mathbf{X}_{\mathbf{k}_{\triangleleft}} = (\mathbf{X} \circ \flat_u)_{k_{\triangleleft}}$ and $\mathbf{X}^{\mathbf{k}^{\triangleright}} = \otimes(\mathbf{X} \circ \mathbf{k}^{\triangleright}) = \otimes(\mathbf{X} \circ \flat_u \circ k^{\triangleright}) = (\mathbf{X} \circ \flat_u)^{k^{\triangleright}}$. Hence $\mathbf{p}_\mathbf{k} \in \mathrm{Hom}(\mathbf{X}^{\mathbf{k}^{\triangleright}}, \mathbf{X}_{\mathbf{k}_{\triangleleft}})$ if and only if $(\mathbf{p} \circ \flat'_u)_k \in \mathrm{Hom}((\mathbf{X} \circ \flat_u)^{k^{\triangleright}}, (\mathbf{X} \circ \flat_u)_{k_{\triangleleft}})$ and this, together with the fact that $\bigcup_{u \in \mathbf{B}^A} \Im(\flat'_u) = \mathbf{K}$, gives the result. $\square$

Let $(\mathbf{X}, \mathbf{p})$ be a $\mathbf{\Lambda}(\mathsf{C})$-algebra. Note then that for each $u \in \mathbf{B}^A$ the $\Lambda(\mathsf{C})$-algebra $(\mathbf{X} \circ \flat_u, \mathbf{p} \circ \flat'_u)$ is bound to the A-family of objects $\mathbf{X} \circ u$.

Proposition 6.2.5 *Let* $(\mathbf{X}, \mathbf{p})$ *and* $(\mathbf{Y}, \mathbf{q})$ *be* $\mathbf{\Lambda}(\mathsf{C})$*-algebras and let* $\pi : \mathbf{X} \to \mathbf{Y}$ *be a* $\mathbf{B}$*-family of morphisms. Then* π *is a* C'*-homomorphism from* $(\mathbf{X}, \mathbf{p})$ *to* $(\mathbf{Y}, \mathbf{q})$ *if and only if* $\pi \circ \flat_u$ *is a* C'*-homomorphism from* $(\mathbf{X} \circ \flat_u, \mathbf{p} \circ \flat'_u)$ *to* $(\mathbf{Y} \circ \flat_u, \mathbf{q} \circ \flat'_u)$ *for each* $u \in \mathbf{B}^A$*.*

Proof This is the same as the proof of Proposition 6.2.4. Let $k \in K$, $u \in \mathbf{B}^A$ and put $\mathbf{k} = \flat'_u(k)$. Then $\mathbf{k}_{\triangleleft} = \flat_u(k_{\triangleleft})$, $\mathbf{k}^{\triangleright} = \flat_u \circ k^{\triangleright}$ and thus $\pi_{\mathbf{k}_{\triangleleft}} = (\pi \circ \flat_u)_{k_{\triangleleft}}$ and $\pi^{\mathbf{k}^{\triangleright}} = \otimes(\pi \circ \mathbf{k}^{\triangleright}) = \otimes(\pi \circ \flat_u \circ k^{\triangleright}) = (\pi \circ \flat_u)^{k^{\triangleright}}$. Hence $\mathbf{q}_\mathbf{k} \circ \pi^{\mathbf{k}^{\triangleright}} = \pi_{\mathbf{k}_{\triangleleft}} \circ \mathbf{p}_\mathbf{k}$ if and only if $(\mathbf{q} \circ \flat'_u)_k \circ (\pi \circ \flat_u)^{k^{\triangleright}} = (\pi \circ \flat_u)_{k_{\triangleleft}} \circ (\mathbf{p} \circ \flat'_u)_k$ and this, together with the fact that $\bigcup_{u \in \mathbf{B}^A} \Im(\flat'_u) = \mathbf{K}$, gives the result. $\square$

We end the section by explicitly constructing an initial term $\mathbf{\Lambda}$-algebra. By Proposition 2.7.5 this just amounts to choosing a suitable locally injective term algebra specifier $\Gamma' : \mathbf{K} \to \Omega$ and a suitable family of enumerations $i'_\mathbf{K}$. In fact there is a surprisingly simple way of making these choices, which will now be explained.

This starts out as if the interest were in the signature Λ (and not $\mathbf{\Lambda}$): For each $k \in K$ choose a bijective mapping i_k from $[n_k]$ to the set $\langle k^{\triangleright} \rangle$, where of course n_k is the cardinality of $\langle k^{\triangleright} \rangle$. Also fix a mapping $\Gamma : K \to \Omega$ (which could also be thought of as a term algebra specifier).

By Lemma 6.2.7 (2) there is a unique mapping $\omega : \mathbf{K} \to K$ with $\omega \circ \flat'_u = \mathrm{id}_K$ for each $u \in \mathbf{B}^A$. Moreover, ω is surjective and for each $\mathbf{k} \in \mathbf{K}$ there exists $u \in \mathbf{B}^A$

such that $\mathbf{k} = \flat'_u(\omega(\mathbf{k}))$. In particular $\langle \mathbf{k}^\triangleright \rangle = \langle \omega(\mathbf{k})^\triangleright \rangle$ for each $\mathbf{k} \in \mathbf{K}$ (since if $u \in \mathbf{B}^A$ is such that $\mathbf{k} = \flat'_u(\omega(\mathbf{k}))$ then $\mathbf{k}^\triangleright = \flat'_u \circ \omega(\mathbf{k})^\triangleright)$. Let $\mathbf{\Gamma} : \mathbf{K} \to \Omega$ be given by $\mathbf{\Gamma} = \Gamma \circ \omega$. Then, since $\langle \mathbf{k}^\triangleright \rangle = \langle \omega(\mathbf{k})^\triangleright \rangle$, a family of enumerations $i'_{\mathbf{K}}$ can be defined by letting $i'_{\mathbf{k}} = i_{\omega(\mathbf{k})}$ for each $\mathbf{k} \in \mathbf{K}$.

Now let $(E_{\mathbf{B}}, \square_{\mathbf{K}})$ be the term $\mathbf{\Lambda}$-algebra specified by $\mathbf{\Gamma}$ and the family $i'_{\mathbf{K}}$. This means that $E_{\mathbf{b}} \subset \Omega^*$ for each $\mathbf{b} \in \mathbf{B}$ and the mapping $\square_{\mathbf{k}} : E^{\mathbf{k}^\triangleright} \to E_{\mathbf{k}_\triangleleft}$ is given for each $\mathbf{k} \in \mathbf{K}$ by

$$\square_{\mathbf{k}}(u) = \Gamma(\omega(\mathbf{k}))\ u(i'_{\mathbf{k}}(1))\ \cdots\ u(i'_{\mathbf{k}}(n_{\mathbf{k}}))$$

for each $u \in E^{\mathbf{k}^\triangleright}$. Moreover, the family $E_{\mathbf{B}}$ can be regarded as being defined by the following rules:

(1) If $\mathbf{k} \in \mathbf{K}$ with $\mathbf{k}^\triangleright = \varepsilon$ then the list consisting of the single component $\Gamma(\omega(\mathbf{k}))$ is an element of $E_{\mathbf{k}_\triangleleft}$.

(2) If $\mathbf{k} \in \mathbf{K}$ with $\mathbf{k}^\triangleright \neq \varepsilon$ and $e_j \in E_{\mathbf{b}_j}$ for $j = 1, \ldots, n_{\mathbf{k}}$, where $\mathbf{b}_j = \mathbf{k}^\triangleright i'_{\mathbf{k}}(j)$, then $\Gamma(\omega(\mathbf{k}))\ e_1\ \cdots\ e_{n_{\mathbf{k}}}$ is an element of $E_{\mathbf{k}_\triangleleft}$.

(3) The only elements in $E_{\mathbf{b}}$ are those which can be obtained using (1) and (2).

Lemma 6.2.8 *If Γ is locally injective then so is $\mathbf{\Gamma}$.*

Proof Since $\mathbf{K}_t = \varnothing$ for each $t \in T$ it is enough to show that the restriction of $\mathbf{\Gamma}$ to $\mathbf{K}_{\mathbf{b}}$ is injective for each $\mathbf{b} \in \mathbf{B} \setminus T$. Thus consider $\mathbf{b} \in \mathbf{B} \setminus T$ and $\mathbf{k}, \mathbf{k}' \in \mathbf{K}_{\mathbf{b}}$ with $\mathbf{\Gamma}(\mathbf{k}) = \mathbf{\Gamma}(\mathbf{k}')$. Now by Lemma 6.2.7 (1) there exists $b \in B \setminus A$ and $u \in \mathbf{B}^A$ such that $\mathbf{b} = \flat_u(b)$, and then by Proposition 6.2.3 (1) there exist $k, k' \in K_b$ with $\mathbf{k} = \flat'_u(k)$ and $\mathbf{k}' = \flat'_u(k')$. Therefore

$$\Gamma(k) = \Gamma(\omega(\flat'_u(k))) = \mathbf{\Gamma}(\mathbf{k}) = \mathbf{\Gamma}(\mathbf{k}') = \Gamma(\omega(\flat'_u(k'))) = \Gamma(k')$$

and so $k = k'$, since the restriction of Γ to K_b is injective. But this implies that $\mathbf{k} = \mathbf{k}'$, i.e., the restriction of $\mathbf{\Gamma}$ to $\mathbf{K}_{\mathbf{b}}$ is injective. $\square$

Proposition 6.2.6 *If Γ is locally injective then $(E_{\mathbf{B}}, \square_{\mathbf{K}})$ is an initial $\mathbf{\Lambda}$-algebra.*

Proof This follows immediately from Proposition 2.7.5 and Lemma 6.2.8. $\square$

6.3 Parameterised algebras

A mapping $\Phi : \mathcal{C}^A \to \Lambda(\mathsf{C})$ is said to be a *parameterised* $\Lambda(\mathsf{C})$*-algebra* if Φ_V is a $\Lambda(\mathsf{C})$-algebra bound to V to each family $V : A \to \mathcal{C}$ (and note that we write Φ_V here instead of $\Phi(V)$).

Of course, if Λ is closed (i.e., if $A = \varnothing$) then $\mathcal{C}^A = \mathbb{I}$ and hence a parameterised $\Lambda(\mathsf{C})$-algebra is then nothing but a single $\Lambda(\mathsf{C})$-algebra. The constructions in this section are thus really meant for open signatures.

Let H be a full subcategory of $\Lambda(\mathsf{C}, \mathsf{C}')$. A parameterised $\Lambda(\mathsf{C})$-algebra Φ is said to be a *parameterised* H*-algebra* if Φ_V is an H-algebra for each $V \in \mathcal{C}^A$ and then a parameterised H-algebra is said to be *intrinsically free* if Φ_V is an intrinsically free H-algebra for each $V \in \mathcal{C}^A$. Thus by definition there exists such a parameterised H-algebra if and only if H possesses intrinsically free objects, and in all the cases we are interested in this requirement is met. However, this is not the end of the story since there is an additional condition which needs to be satisfied and which has to do with how Φ_V behaves as a function of V. How this condition arises and the problem of constructing parameterised algebras satisfying it is the topic to be discussed in the present section.

Let Φ be a parameterised $\Lambda(\mathsf{C})$-algebra with $\Phi = (X, p)$; thus $X : \mathcal{C}^A \to \mathcal{C}^B$ and $p : \mathcal{C}^A \to \mathcal{M}^K$, and we write (X^V, p^V) instead of (X_V, p_V) for the $\Lambda(\mathsf{C})$-algebra Φ_V. Let $\lfloor\cdot\rfloor$ be a support for Λ. A parameterised $\Lambda(\mathsf{C})$-algebra $\Phi : \mathcal{C}^A \to \Lambda(\mathsf{C})$ is said to be *compatible* with $\lfloor \cdot \rfloor$ if:

(1) $X^V_b = X^W_b$ whenever $V_a = W_a$ for all $a \in \lfloor b \rfloor$.

(2) $p^V_k = p^W_k$ whenever $V_a = W_a$ for all $a \in \lfloor k_\triangleleft \rfloor$.

Note that if a parameterised algebra is compatible with the minimal support then it is compatible with any support.

Example 6.3.1 Let $\Lambda = (B, K, \Theta)$ be the signature in Example 6.3.1 considered the with the minimal support $\lfloor\cdot\rfloor$.

For each $V : A \to \mathsf{BSets}$ denote by Φ_V the bottomed Λ-algebra defined in Example 3.1.3 (and denoted there by (Y, q)); thus Φ_V is bound to V. This gives a parameterised $\Lambda(\mathsf{BSets})$-algebra Φ which is compatible with $\lfloor\cdot\rfloor$.

In what follows we will need parameterised algebras which are compatible with a given support $\lfloor\cdot\rfloor$, and in general it is not even clear whether such an algebra

exists. In particular, there seems little hope of obtaining them using something like Proposition 3.3.1, Proposition 5.1.3 or Proposition 5.2.2: The fact that a $\Lambda(\mathsf{C})$-algebra Φ_V bound to V is unique up to an appropriate isomorphism for each V does not say anything about how Φ_V behaves as a function of V. This suggests that, in order to obtain parameterised algebras compatible with $\lfloor\cdot\rfloor$, it is necessary to give an absolutely explicit method of constructing certain classes of algebras. No doubt this is possible, but it would be rather involved. We thus prefer to take an easy way out, which at first sight seems somewhat restrictive, but turns out not to be so.

The starting point for our approach is contained in the following trivial fact:

Lemma 6.3.1 *If A is finite then any parameterised $\Lambda(\mathsf{C})$-algebra $\Phi : \mathcal{C}^A \to \Lambda(\mathsf{C})$ is compatible with the maximal support $\lfloor\cdot\rfloor^* : B \to \mathcal{P}_o(A)$ (given by $\lfloor b\rfloor^* = A$ for each $b \in B \setminus A$ and $\lfloor a\rfloor^* = \{a\}$ for each $a \in A$).*

Proof Let $\Phi = (X, p)$. Let $b \in B$ and $V, W \in \mathcal{C}^A$ with $V_a = W_a$ for all $a \in \lfloor b\rfloor^*$. If $b \in B \setminus A$ then $\lfloor b\rfloor^* = A$, thus $V = W$ and hence $X_b^V = X_b^W$. On the other hand, if $b \in A$ then $\lfloor b\rfloor^* = \{b\}$, thus $V_b = W_b$ and again $X_b^V = V_b = W_b = X_b^W$. This shows that (1) holds, and (2) follows in the same way, since if $k \in K$ then $k_\triangleleft \in B \setminus A$ and so $\lfloor k_\triangleleft\rfloor^* = A$. $\square$

The signature Λ will be called *simple* if A is finite and there is only one support for Λ (i.e., if the maximal support is also the minimal support). If Λ is simple then Lemma 6.3.1 says that any parameterised $\Lambda(\mathsf{C})$-algebra is compatible with the minimal support (and thus is compatible with any support). Of course, Λ will rarely be simple and so this fact cannot be applied directly. However, it can be applied indirectly whenever Λ is the disjoint union of simple signatures.

In what follows suppose Λ is the disjoint union of the signatures $\Lambda_i = (B_i, K_i, \Theta_i)$, $i \in F$; thus $A = \bigcup_{i\in F} A_i$, with A_i the parameter set of Λ_i. For each $i \in F$ let $\Phi^i : \mathcal{C}^{A_i} \to \Lambda_i(\mathsf{C})$ be a parameterised $\Lambda_i(\mathsf{C})$-algebra, and define $\Phi : \mathcal{C}^A \to \Lambda(\mathsf{C})$ by putting $\Phi_V = \oplus_{i\in F}\Phi^i_{V_i}$ for all $V \in \mathcal{C}^A$, where V_i is the restriction of V to A_i. Thus Φ is a parameterised $\Lambda(\mathsf{C})$-algebra, which will be called the *sum* of the parameterised algebras Φ^i, $i \in F$, and will be denoted by $\oplus_{i\in F}\Phi^i$.

Proposition 6.3.1 *If each of the signatures Λ_i, $i \in F$, is simple then any sum $\oplus_{i\in F}\Phi^i$ of parameterised $\Lambda_i(\mathsf{C})$-algebras is compatible with the minimal support for Λ.*

Proof For each $i \in F$ let $\Phi^i = (X^i, p^i)$, let $\lfloor\cdot\rfloor_i$ be the unique support for Λ_i and define a mapping $\lfloor\cdot\rfloor : B \to \mathcal{P}_o(A)$ by letting $\lfloor b\rfloor = \lfloor b\rfloor_i$ for each $b \in B_i$; then $\lfloor\cdot\rfloor$ is a support for Λ and it is clear that $\lfloor\cdot\rfloor$ is in fact the minimal support. Consider

$b \in B$ and $V, W \in \mathcal{C}^A$ with $V_a = W_a$ for all $a \in \lfloor b \rfloor$; let i be such that $b \in B_i$. Then $\lfloor b \rfloor_i = \lfloor b \rfloor$ and $(V_i)_a = V_a$, $(W_i)_a = W_a$ for all $a \in \lfloor b \rfloor_i$ (since $\lfloor b \rfloor_i \subset A_i$). Thus $X_b^V = (X^i)_b^{V_i} = (X^i)_b^{W_i} = X_b^W$, because by Lemma 6.3.1 Φ^i is compatible with $\lfloor \cdot \rfloor_i$. In the same way $p_k^V = p_k^W$ whenever $V_a = W_a$ for all $a \in \lfloor k_\triangleleft \rfloor$, and therefore the sum $\oplus_{i \in F} \Phi^i$ is compatible with $\lfloor \cdot \rfloor$. $\square$

Proposition 6.3.2 *Let H_i be a full subcategory of $\Lambda_i(\mathsf{C}, \mathsf{C}')$ which possesses intrinsically free objects for each $i \in F$ and let $\mathsf{H} = \oplus_{i \in F} \mathsf{H}_i$. If each Λ_i is simple then there exists an intrinsically free parameterised H-algebra compatible with the minimal support for Λ.*

Proof There exists an intrinsically free parameterised H_i-algebra for each $i \in F$ and by Proposition 6.3.1 their sum is a parameterised $\Lambda(\mathsf{C})$-algebra which is compatible with the minimal support for Λ. Moreover, by Proposition 6.1.2 this sum is also H-intrinsically free. $\square$

Let us now say that the signature Λ is *semi-simple* if it can be obtained as the disjoint union of simple signatures Λ_i, $i \in F$.

Proposition 6.3.3 *Suppose the signature Λ is semi-simple. Then there exists an intrinsically free parameterised H-algebra compatible with the minimal support for Λ in each of the following three cases, in each case with $\diamond$ a $\natural$-stable simple head type and with C' the subcategory defined by requiring the morphisms to be bottomed:*

(1) $\mathsf{C} = \mathsf{BSets}$ *with the objects of* H *the bottomed $\diamond$-algebras.*

(2) $\mathsf{C} = \mathsf{Posets}$ *with the objects of* H *the ordered $\diamond$-algebras.*

(3) $\mathsf{C} = \mathsf{APosets}$ *with the objects of* H *the continuous $\diamond$-algebras.*

Proof These follow immediately from Proposition 6.3.2 and Lemma 6.1.4. $\square$

If Λ is semi-simple then the results of this section imply that in the cases we are interested in there exist initial parameterised H-algebras compatible with the minimal support for Λ. The problem is now, of course, that a typical open signature need not be semi-simple, and in fact this is the case with the signature in Example 2.2.3. However, it is always possible to replace a given signature with a semi-simple signature (with a larger parameter set) which is, in a certain sense, more general. This is illustrated in Example 6.3.2, where a semi-simple replacement for the signature in Example 2.2.3 is presented. (It is left for the reader to to work out how to make this replacement in the general case.)

Example 6.3.2 Consider the signature $\Lambda = (B, K, \Theta)$ with parameter set $A = \{\mathtt{x}, \mathtt{y}, \mathtt{z}, \mathtt{u}, \mathtt{v}\}$ given by

$B = \{\mathtt{bool}, \mathtt{atom}, \mathtt{int}, \mathtt{pair}, \mathtt{list}, \mathtt{lp}, \mathtt{x}, \mathtt{y}, \mathtt{z}, \mathtt{u}, \mathtt{v}\}$,

$K = \{\mathtt{True}, \mathtt{False}, \mathtt{Atom}, \mathtt{Pair}, \mathtt{Nil}, \mathtt{Cons}, \mathtt{L}, \mathtt{P}\} \cup \underline{Int}$,

and with $\Theta : K \to B^* \times B$ defined by

$\Theta(\mathtt{True}) = \Theta(\mathtt{False}) = (\varepsilon, \mathtt{bool})$,

$\Theta(\mathtt{Atom}) = (\varepsilon, \mathtt{atom})$, $\Theta(\mathtt{Pair}) = (\mathtt{x}\ \mathtt{y}, \mathtt{pair})$,

$\Theta(\mathtt{Nil}) = (\varepsilon, \mathtt{list})$, $\Theta(\mathtt{Cons}) = (\mathtt{z}\ \mathtt{list}, \mathtt{list})$,

$\Theta(\mathtt{L}) = (\mathtt{u}, \mathtt{lp})$, $\Theta(\mathtt{P}) = (\mathtt{v}, \mathtt{lp})$,

$\Theta(\underline{n}) = (\varepsilon, \mathtt{int})$ for each $n \in \mathbb{Z}$,

and which can be represented in the (augmented) form

$$\begin{array}{l}
\mathtt{bool} ::= \mathtt{True} \mid \mathtt{False} \\
\mathtt{atom} ::= \mathtt{Atom} \\
\mathtt{int} ::= \cdots -2 \mid -1 \mid 0 \mid 1 \mid 2 \ \cdots \\
(\mathtt{pair}\ \mathtt{x}\ \mathtt{y}) ::= \mathtt{Pair}\ \mathtt{x}\ \mathtt{y} \\
(\mathtt{list}\ \mathtt{z}) ::= \mathtt{Nil} \mid \mathtt{Cons}\ \mathtt{z}\ (\mathtt{list}\ \mathtt{z}) \\
(\mathtt{lp}\ \mathtt{u}\ \mathtt{v}) ::= \mathtt{L}\ \mathtt{u} \mid \mathtt{P}\ \mathtt{v}
\end{array}$$

This signature Λ is easily seen to be semi-simple.

6.4 Polymorphic algebras

Recall once again we have fixed a support $\lfloor\cdot\rfloor : B \to \mathcal{P}_o(A)$ for Λ and a set of type variables T. A T-free set of polymorphic types $(\mathbf{B}, \#)$ and an initial set of polymorphic operator names $(\mathbf{K}, \#')$ were then fixed. We are assuming that $\mathbf{B} \neq \varnothing$ and so these objects give rise to the polymorphic signature $\mathbf{\Lambda} = (\mathbf{B}, \mathbf{K}, \mathbf{\Theta})$.

If $\mathbf{\Phi} = (\mathbf{X}, \mathbf{p}) : \mathcal{C}^T \to \mathbf{\Lambda}(\mathsf{C})$ is a parameterised $\mathbf{\Lambda}(\mathsf{C})$-algebra then for each $U \in \mathcal{C}^T$, $u \in \mathbf{B}^A$, the $\Lambda(\mathsf{C})$-algebra $(\mathbf{X}^U \circ \flat_u, \mathbf{p}^U \circ \flat'_u)$ will be denoted by $\mathbf{\Phi}_U \circ \flat^o_u$.

Proposition 6.4.1 *Let $\Phi : \mathcal{C}^A \to \Lambda(\mathsf{C})$ be a parameterised $\Lambda(\mathsf{C})$-algebra which is compatible with $\lfloor\cdot\rfloor$. Then there exists a unique parameterised $\mathbf{\Lambda}(\mathsf{C})$-algebra $\mathbf{\Phi} = (\mathbf{X}, \mathbf{p}) : \mathcal{C}^T \to \mathbf{\Lambda}(\mathsf{C})$ such that*

$$\mathbf{\Phi}_U \circ \flat^o_u = \Phi_V$$

with $V = \mathbf{X}^U \circ u$ for all $U \in \mathcal{C}^T$, $u \in \mathbf{B}^A$.

Proof Define a mapping $\varphi : \mathcal{C}^A \to \mathcal{C}^B$ by $\varphi(V)(b) = X^V_b$ for all $V \in \mathcal{C}^A$, $b \in B$, where $\Phi = (X, p)$; thus in particular $\varphi(V)(b) = X^V_a = V(a)$ for all $V \in \mathcal{C}^A$,

$a \in A$. Moreover, φ is B-compatible with $\lfloor\cdot\rfloor$, since Φ is compatible with $\lfloor\cdot\rfloor$. By Proposition 6.2.3 (1) there thus exists for each $U \in \mathcal{C}^T$ a unique mapping $\mathbf{X}^U : \mathbf{B} \to \mathcal{C}$ with ${\mathbf{X}^U}_{|T} = U$ such that $\mathbf{X}^U \circ \flat_u = \varphi(\mathbf{X}^U \circ u)$ for all $u \in \mathbf{B}^A$. It then follows that $\mathbf{X}^U \circ \flat_u = X^V$ with $V = \mathbf{X}^U \circ u$ for each $u \in \mathbf{B}^A$, because $\varphi(\mathbf{X}^U \circ u)(b) = X^V_b$ for each $b \in B$.

Now for each $U \in \mathcal{C}^T$ define $\psi^U : \mathbf{B}^A \to \mathcal{M}^K$ by $\psi^U(u)(k) = p^V_k$ for all $u \in \mathbf{B}^A$, $k \in K$, where $V = \mathbf{X}^U \circ u$. Then ψ^U is K-compatible with $\lfloor\cdot\rfloor$, since Φ is compatible with $\lfloor\cdot\rfloor$. By Proposition 6.2.3 (2) there thus exists a unique mapping $\mathbf{p}^U : \mathbf{K} \to \mathcal{M}$ such that $\mathbf{p}^U \circ \flat'_u = \psi^U(u)$ for all $u \in \mathbf{B}^A$. Therefore $\mathbf{p}^U \circ \flat'_u = p^V$ with $V = \mathbf{X}^U \circ u$ for all $u \in \mathbf{B}^A$.

By Proposition 6.2.4 $(\mathbf{X}^U, \mathbf{p}^U)$ is a $\mathbf{\Lambda}(\mathsf{C})$-algebra for each $U \in \mathcal{C}^T$ and $(\mathbf{X}^U, \mathbf{p}^U)$ is bound to U, since by definition ${\mathbf{X}^U}_{|T} = U$. A parameterised $\mathbf{\Lambda}(\mathsf{C})$-algebra $\mathbf{\Phi} : \mathcal{C}^T \to \mathbf{\Lambda}(\mathsf{C})$ can thus be defined by letting $\mathbf{\Phi}_U = (\mathbf{X}^U, \mathbf{p}^U)$ for each $U \in \mathcal{C}^T$ and by construction $\mathbf{\Phi}_U \circ \flat^o_u = \Phi_V$ with $V = \mathbf{X}^U \circ u$ for all $U \in \mathcal{C}^T$, $u \in \mathbf{B}^A$. The uniqueness follows from the uniqueness in Proposition 6.2.1. □

The parameterised $\mathbf{\Lambda}(\mathsf{C})$-algebra $\mathbf{\Phi}$ given in Proposition 6.4.1 will be called the *polymorphic* $\mathbf{\Lambda}(\mathsf{C})$*-algebra* corresponding to the parameterised $\Lambda(\mathsf{C})$-algebra Φ.

The final task is to show that the polymorphic algebra $\mathbf{\Phi}$ inherits the properties of Φ which are needed in applications. In particular, it will be established that the $\mathbf{\Lambda}(\mathsf{C})$-algebra $\mathbf{\Phi}_U$ is regular for each $U \in \mathcal{C}^T$ if the $\Lambda(\mathsf{C})$-algebra Φ_V is regular for each $V \in \mathcal{C}^A$. We will employ a set-up here (using what we call a reduction) which allows things to be formulated not just for a particular category such as BSets.

Let D be a further category. A mapping $\Delta : \mathcal{C}_\mathsf{C} \to \mathcal{C}_\mathsf{D}$ will be called a *reduction* from C to D if

— $\mathrm{Hom}_\mathsf{C}(X, Y) \subset \mathrm{Hom}_\mathsf{D}(\Delta(X), \Delta(Y))$ for all objects X, $Y \in \mathcal{C}_\mathsf{C}$,

— $g \circ_\mathsf{C} f = g \circ_\mathsf{D} f$ for all $f \in \mathrm{Hom}_\mathsf{C}(X, Y)$, $g \in \mathrm{Hom}_\mathsf{C}(Y, Z)$ for all objects X, Y, $Z \in \mathcal{C}_\mathsf{C}$,

— the identity morphism in $\mathrm{Hom}_\mathsf{C}(X, X)$ is equal to the identity morphism in $\mathrm{Hom}_{\mathsf{C}'}(\Delta(X), \Delta(X))$ for each object $X \in \mathcal{C}_\mathsf{C}$,

(where the subscripts C and D indicate which category is involved). If D (as well as C) is a $\otimes$-category then a reduction $\Delta : \mathcal{C}_\mathsf{C} \to \mathcal{C}_\mathsf{D}$ is called a $\otimes$*-reduction* if $\Delta(\otimes X) = \otimes(\Delta \circ X)$ for all $X \in \mathcal{F}(\mathcal{C}_\mathsf{C})$.

A reduction should be thought of as as an operation which forgets some of the structure. There are obvious $\otimes$-reductions from $\mathsf{APosets}$ and $\mathsf{CPosets}$ to Posets, from Posets to BSets and from BSets to Sets. These will be referred to as the *standard* $\otimes$*-reductions*.

The category C together with a reduction from C to Sets is usually called a *concrete category*.

Lemma 6.4.1 *Suppose* D *is a* $\otimes$*-category and* Δ *is a* $\otimes$*-reduction from* C *to* D*. Then* $(\Delta \circ X, p)$ *is a* $\Lambda(\mathsf{D})$*-algebra for each* $\Lambda(\mathsf{C})$*-algebra* (X, p)*. Moreover, if* (X, p) *is bound to the family* $V : A \to \mathcal{C}_{\mathsf{C}}$ *then* $(\Delta \circ X, p)$ *is bound to the family* $\Delta \circ V : A \to \mathcal{C}_{\mathsf{D}}$*.*

Proof Let $k \in K$; then $p_k \in \mathrm{Hom}_{\mathsf{C}}(X^{k^\triangleright}, X_{k_\triangleleft}) \subset \mathrm{Hom}_{\mathsf{D}}(\Delta(X^{k^\triangleright}), \Delta(X_{k_\triangleleft}))$ and thus $p_k \in \mathrm{Hom}_{\mathsf{D}}((\Delta \circ X)^{k^\triangleright}, (\Delta \circ X)_{k_\triangleleft})$, since

$$(\Delta \circ X)^{k^\triangleright} = \otimes(\Delta \circ X \circ k^\triangleright) = \Delta(\otimes(X \circ k^\triangleright)) = \Delta(X^{k^\triangleright}) \ .$$

This implies $(\Delta \circ X, p)$ is a $\Lambda(\mathsf{D})$-algebra. The final statement is clear. $\square$

The $\Lambda(\mathsf{D})$-algebra $(\Delta \circ X, p)$ in Lemma 6.4.1 will be denoted by $\Delta \circ (X, p)$. Lemma 6.4.1 allows some of the definitions introduced in the previous chapters to be reformulated in the following style:

Let C be one of the categories $\mathsf{APosets}$, $\mathsf{CPosets}$ and Posets and let Δ be the standard reduction from C to BSets. If $(H, \diamond)$ is a head type then a $\Lambda(\mathsf{C})$-algebra (X, p) was defined to be an $(H, \diamond)$-algebra if the bottomed Λ-algebra $\Delta \circ (X, p)$ is an $(H, \diamond)$-algebra. Moreover, if (X, p) is bound to the family $V : A \to \mathcal{C}$ then (X, p) was defined to be V-minimal if $\Delta \circ (X, p)$ is $\Delta \circ V$-minimal.

In the same way, let C be one of the categories $\mathsf{APosets}$ and $\mathsf{CPosets}$ and let Δ be the standard reduction from C to Posets (so in fact Δ is just the inclusion mapping). Then a $\Lambda(\mathsf{C})$-algebra (X, p) was defined to be intrinsic if the ordered Λ-algebra $\Delta \circ (X, p)$ is intrinsic.

In what follows let $\Phi = (X, p)$ be a parameterised $\Lambda(\mathsf{C})$-algebra compatible with $\lfloor\cdot\rfloor$ and let $\boldsymbol{\Phi} = (\mathbf{X}, \mathbf{p})$ be the corresponding polymorphic $\boldsymbol{\Lambda}(\mathsf{C})$-algebra.

Proposition 6.4.2 *Let* Δ *be a* $\otimes$*-reduction from* C *to* BSets *and suppose that* $\Delta \circ \Phi_V$ *is a regular bottomed* Λ*-algebra for each* V*. Then* $\Delta \circ \boldsymbol{\Phi}_U$ *is a regular bottomed* $\boldsymbol{\Lambda}$*-algebra for each* U*.*

Proof Put $\Delta \circ X^V = \breve{X}^V$ for each $V \in \mathcal{C}^A$ and $\Delta \circ \mathbf{X}^U = \breve{\mathbf{X}}^U$ for each $U \in \mathcal{C}^T$. Let $U \in \mathcal{C}^T$, $\mathbf{b} \in \mathbf{B} \setminus T$ and $x \in (\breve{\mathbf{X}}^U_{\mathbf{b}})^\natural$; it must then be shown that there exists a unique $\mathbf{k} \in \mathbf{K}_{\mathbf{b}}$ and a unique $v \in \mathrm{dom}(\mathbf{p}^U_{\mathbf{k}})$ such that $x = \mathbf{p}^U_{\mathbf{k}}(v)$. Now by Lemma 6.2.7 (2) there exists $b \in B \setminus A$ and $u \in \mathbf{B}^{\lfloor b \rfloor}$ with $\mathbf{b} = \flat_u(b)$, and then by Proposition 6.2.3 (1) $\mathbf{K}_{\mathbf{b}} = \{\flat'_u(k) : k \in K_b\}$. Let $V = \mathbf{X}^U \circ u$. Then $\mathbf{X}^U_{\mathbf{b}} = X^V_b$, which means also that $\breve{\mathbf{X}}^U_{\mathbf{b}} = \breve{X}^V_b$, and $(\mathbf{p}^U \circ \flat'_u)_k = p^V_k$ for each $k \in K_b$. But $(\breve{X}^V, p^V)$ is a regular bottomed Λ-algebra, $b \in B \setminus A$ and $x \in (\breve{X}^V_b)^\natural$. There thus exists a unique $k \in K_b$ and a unique $v \in \mathrm{dom}(p^V_k)$ such that $x = p^V_k(v)$. Hence, putting $\mathbf{k} = \flat'_u(k)$, it follows that $\mathbf{k} \in \mathbf{K}_{\mathbf{b}}$ and $x = \mathbf{p}^U_{\mathbf{k}}(v)$. Conversely, if $x = \mathbf{p}^U_{\mathbf{k}'}(v')$ with $\mathbf{k}' \in \mathbf{K}_{\mathbf{b}}$ and $v' \in \mathrm{dom}(\mathbf{p}^U_{\mathbf{k}'})$ then $\mathbf{k}' = \flat'_u(k')$ for some $k' \in K_b$,

and so $p_k^V(v) = x = \mathbf{p}_{\mathbf{k}'}^U(v') = p_{k'}^V(v')$, which implies that $k' = k$, thus also that $\mathbf{k}' = \mathbf{k}$, and that $v' = v$ (since $(\breve{X}^V, p^V)$ is regular). This shows $\Delta \circ \mathbf{\Phi}_U$ is a regular bottomed $\mathbf{\Lambda}$-algebra. $\square$

Let $(H, \diamond)$ be a simple head type for Λ (so $H_b = \mathbb{T}$ for each $b \in B$); then a simple head type $(\mathbf{H}, \diamond')$ for the signature $\mathbf{\Lambda}$ (so again $\mathbf{H_b} = \mathbb{T}$ for each $\mathbf{b} \in \mathbf{B}$) can be defined as follows: Recall that there is a unique mapping $\omega : \mathbf{K} \to K$ with $\omega \circ \flat'_u = \mathrm{id}_K$ for each $u \in \mathbf{B}^A$, and that $\langle \mathbf{k}^\triangleright \rangle = \langle \omega(\mathbf{k})^\triangleright \rangle$ for each $\mathbf{k} \in \mathbf{K}$. Consider $\mathbf{k} \in \mathbf{K}$ and put $k = \omega(\mathbf{k})$; Then $\mathbf{H}^{\mathbf{k}^\triangleright} = \mathbb{T}^{\langle \mathbf{k}^\triangleright \rangle} = \mathbb{T}^{\langle k^\triangleright \rangle} = H^{k^\triangleright}$ and $\mathbf{H}_{\mathbf{k}_\triangleleft} = \mathbb{T} = H_{k_\triangleleft}$, and so a mapping $\diamond'_{\mathbf{k}} : \mathbf{H}^{\mathbf{k}^\triangleright} \to \mathbf{H_b}$ can be defined by putting $\diamond'_{\mathbf{k}}(v) = \diamond_k(v)$ for all $v \in \mathbf{H}^{\mathbf{k}^\triangleright}$. Thus in fact $\diamond' = \diamond \circ \omega$. If $\diamond$ is $\natural$-invariant (resp. $\natural$-stable) then so is $\diamond'$.

Proposition 6.4.3 *Let Δ be a $\otimes$-reduction from C to BSets and suppose that $\Delta \circ \Phi_V$ is a $\diamond$-algebra for each V. Then $\Delta \circ \mathbf{\Phi}_U$ is a $\diamond'$-algebra for each U.*

Proof Put $\Delta \circ X^V = \breve{X}^V$ for each $V \in \mathcal{C}^A$ and $\Delta \circ \mathbf{X}^U = \breve{\mathbf{X}}^U$ for each $U \in \mathcal{C}^T$. Let $U \in \mathcal{C}^T$ and $\mathbf{k} \in \mathbf{K}$; by Lemma 6.2.7 (2) there exists $k \in K$ and $u \in \mathbf{B}^A$ such that $\mathbf{k} = \flat'_u(k)$, and $\langle \mathbf{k}^\triangleright \rangle = \langle k^\triangleright \rangle$. Let $V = \mathbf{X}^U \circ u$. Then $\mathbf{X}_{\mathbf{b}}^U = X_b^V$, $(\mathbf{X}^U)^{\mathbf{k}^\triangleright} = (X^V)^{k^\triangleright}$, hence also $\breve{\mathbf{X}}_{\mathbf{b}}^U = \breve{X}_b^V$, $(\breve{\mathbf{X}}^U)^{\mathbf{k}^\triangleright} = (\breve{X}^V)^{k^\triangleright}$, and $\mathbf{p}_{\mathbf{k}}^U = p_k^V$. Therefore $\varepsilon_{\mathbf{k}_\triangleleft} = \varepsilon_{k_\triangleleft}$ (as mappings from $\breve{\mathbf{X}}_{\mathbf{k}_\triangleleft}^U = \breve{X}_{k_\triangleleft}^V$ to $\mathbf{H}_{\mathbf{k}_\triangleleft} = H_{k_\triangleleft}$) and $\varepsilon^{\mathbf{k}^\triangleright} = \varepsilon^{k^\triangleright}$ (as mappings from $(\breve{\mathbf{X}}^U)^{\mathbf{k}^\triangleright} = (\breve{X}^V)^{k^\triangleright}$ to $\mathbf{H}^{\mathbf{k}^\triangleright} = H^{k^\triangleright}$). Moreover, $\diamond'_{\mathbf{k}} = \diamond_k$ holds by definition. But $\varepsilon_{k_\triangleleft} \circ p_k^V = \diamond_k \circ \varepsilon^{k^\triangleright}$, because $(\breve{X}^V, p^V)$ is a $\diamond$-algebra, and so $\varepsilon_{\mathbf{k}_\triangleleft} \circ \mathbf{p}_{\mathbf{k}}^U = \diamond'_{\mathbf{k}} \circ \varepsilon^{\mathbf{k}^\triangleright}$, which implies that $\Delta \circ \mathbf{\Phi}_U$ is a $\diamond'$-algebra. $\square$

Proposition 6.4.4 *Let Δ be a $\otimes$-reduction from C to Posets and suppose that $\Delta \circ \Phi_V$ is an intrinsic ordered Λ-algebra for each V. Then $\Delta \circ \mathbf{\Phi}_U$ is an intrinsic ordered $\mathbf{\Lambda}$-algebra for each U.*

Proof This is very similar to the proof of Proposition 6.4.2 and is left for the reader. $\square$

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